Direction for Exercises 65 to 69:

Verify the Rolle's theorem for each of the functions in these exercises.

65. $f(x) = x(x-1)^2$ in $[0, 1]$.

Sol. We have, $f(x) = x(x-1)^2$ in $[0, 1]$

Since, $f(x) = x(x-1)^2$ is a polynomial function it is continuous in $[0, 1]$ and differentiable in $(0, 1)$

Now, $f(0) = 0$ and $f(1) = 0 \Rightarrow f(0) = f(1)$

f satisfies the conditions of Rolle's theorem.

Hence, by Rolle's theorem there exists atleast one $c \in (0, 1)$ such that

$$f'(c) = 0$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow \quad (3c-1)(c-1) = 0$$

$$\Rightarrow \quad c = \frac{1}{3} \in (0, 1)$$

Therefore, Rolle's theorem has been verified.

69. $f(x) = \sqrt{4 - x^2}$ in $[-2, 2]$

Sol. We have, $f(x) = \sqrt{4 - x^2} = (4 - x^2)^{1/2}$

Since $(4 - x^2)$ and square root function are continuous and differentiable in their domain, given function $f(x)$ is also continuous and differentiable in $[-2, 2]$

Also $f(-2) = f(2) = 0$

So, conditions of Rolle's theorem are satisfied.

Hence, there exists *a* real number $c \in (-2, 2)$ such that $f'(c) = 0$.

Now $$f'(x) = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{4 - x^2}}$$

So, $f'(c) = 0$

$$\Rightarrow \quad -\frac{c}{\sqrt{4 - c^2}} = 0$$

$$\Rightarrow \quad c = 0 \in (-2, 2)$$

Hence, Rolle's theorem has been verified.