Example 18 Verify mean value theorem for the function $f(x) = (x-3)(x-6)(x-9)$ in $[3, 5]$.

Solution (i) Function f is continuous in $[3, 5]$ as product of polynomial functions is a polynomial, which is continuous.

(ii) $f'(x) = 3x^2 - 36x + 99$ exists in $(3, 5)$ and hence derivable in $(3, 5)$.

Thus conditions of mean value theorem are satisfied. Hence, there exists at least one $c \in (3, 5)$ such that

$$f'(c) = \frac{f(5) - f(3)}{5 - 3}$$

$$\Rightarrow 3c^2 - 36c + 99 = \frac{8 - 0}{2} = 4$$

$$\Rightarrow c = 6 \pm \sqrt{\frac{13}{3}}.$$

Hence $c = 6 - \sqrt{\frac{13}{3}}$ (since other value is not permissible).