PROBLEM

Let $g(x) = \log f(x)$ where $f(x)$ is twice differentible positive function on $(0, \infty)$ such that $f(x+1) = x f(x)$. Then, for $N = 1, 2, 3, \ldots\ldots\ldots$ *(2008)*

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(a) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots\ldots+\frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots\ldots+\frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots\ldots+\frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots\ldots+\frac{1}{(2N+1)^2}\right\}$

(a) Given that $g(x) = \log f(x) \Rightarrow g(x+1) = \log f(x+1)$

$\Rightarrow \quad g(x+1) = \log x f(x) \qquad [\because f(x+1) = x f(x)]$

$\Rightarrow \quad g(x+1) = \log x + \log f(x) \Rightarrow g(x+1) - g(x) = \log(x)$

$$\Rightarrow \quad g'(x+1) - g'(x) = \frac{1}{x}$$

$$\Rightarrow \quad g''(x+1) - g''(x) = -\frac{1}{x^2}$$

Putting, $x = x - \frac{1}{2}$, we get

$$\Rightarrow \quad g''\left(x+\frac{1}{2}\right)-g''\left(x-\frac{1}{2}\right)=-\frac{1}{\left(x-\frac{1}{2}\right)^2}=\frac{-2^2}{(2x-1)^2}$$

Putting $x = 1, 2, 3$ ...., N we get

$$g''\left(\frac{3}{2}\right)-g''\left(\frac{1}{2}\right)=-\frac{2^2}{1^2} \qquad ...(1)$$

$$g''\left(\frac{5}{2}\right)-g''\left(\frac{3}{2}\right)=\frac{-2^2}{3^2} \qquad ...(2)$$

$$g''\left(\frac{7}{2}\right)-g''\left(\frac{5}{2}\right)=\frac{-2^2}{5^2} \qquad ...(3)$$

$$g''\left(N+\frac{1}{2}\right)-g''\left(N-\frac{1}{2}\right)=-\frac{2^2}{(2N-1)^2} \qquad ...(N)$$

Adding all the above equations, we get

$$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=-4\left[1+\frac{1}{3^2}+\frac{1}{5^2}+.....+\frac{1}{(2N-1)^2}\right]$$