PROBLEM

If $0 < x_1 < x_2 < x_3 < \pi$, then prove that $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right) > \frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$. Hence or otherwise prove that if A, B, C are angles of a triangle, then the maximum value of $\sin A + \sin B + \sin C$ is $\frac{3\sqrt{3}}{2}$.

SOLUTION

5. Let points A, B, C form a triangle. The y-coordinate of centroid G is $\frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$ and the y-coordinate of point F is $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right)$.

From the figure, $FD > GD$.

Hence, $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right) > \frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$.

If $A + B + C = \pi$, then

$$\sin\left(\frac{A + B + C}{3}\right) > \frac{\sin A + \sin B + \sin C}{3}$$

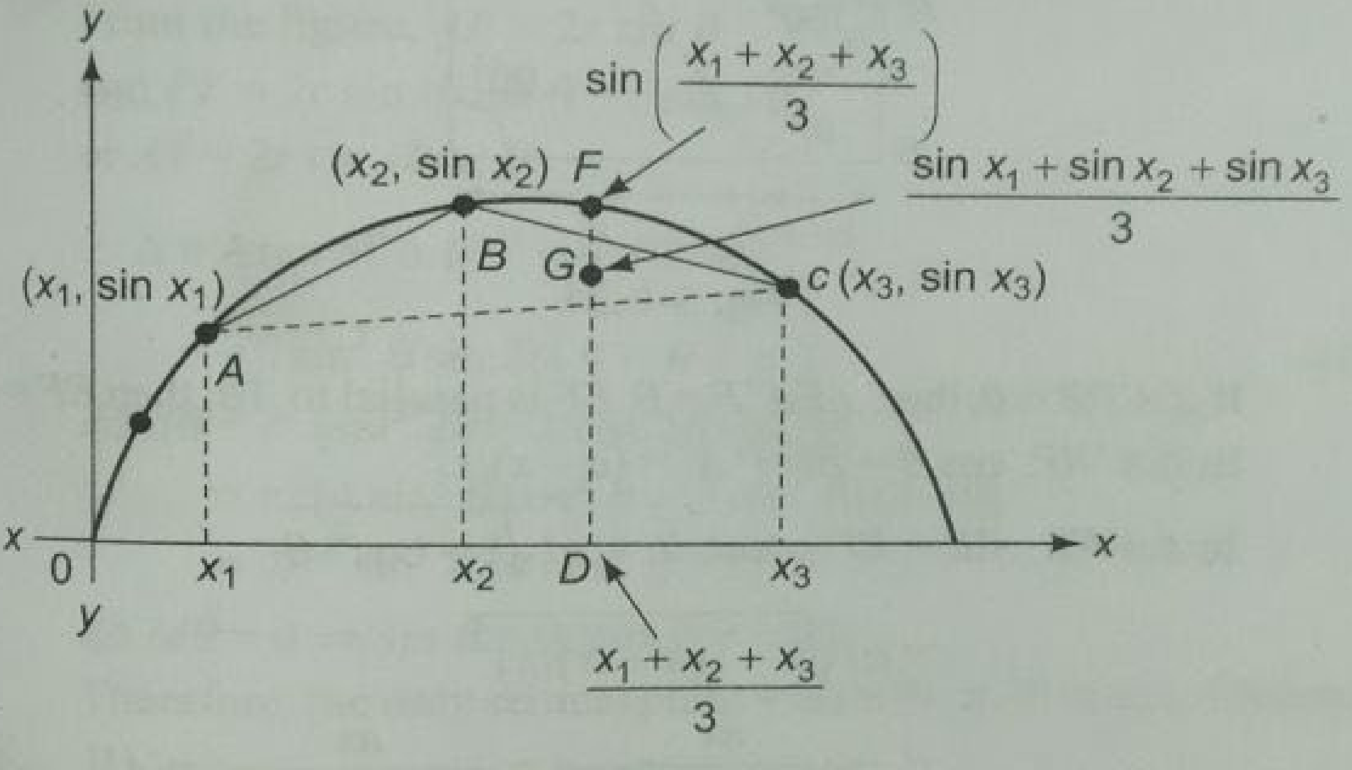

Fig. S-6.19

or $\sin\frac{\pi}{3} > \frac{\sin A + \sin B + \sin C}{3}$

or $\frac{3\sqrt{3}}{2} > \sin A + \sin B + \sin C$

or maximum value of $(\sin A + \sin B + \sin C) = \frac{3\sqrt{3}}{2}$