PROBLEM

Let $p(x)$ be a polynomial of degree 4 having extremum at

$x = 1, 2$ and $\lim_{x \to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2.$

Then the value of $p\,(2)$ is *(2009)*

SOLUTION

(0) Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

Now $\lim_{x \to 0}\left[1 + \frac{p(x)}{x^2}\right] = 2$

$\Rightarrow \lim_{x \to 0} \frac{p(x)}{x^2} = 1$ ...(1)

$\Rightarrow p(0) = 0 \Rightarrow e = 0$

Applying L 'Hospital's rule to eq^n (1), we get

$\lim_{x \to 0} \frac{p'(x)}{2x} = 1 \Rightarrow p'(0) = 0$

$\Rightarrow d = 0$

Again applying L 'Hospital's rule, we get

$\lim_{x \to 0} \frac{p''(x)}{2} = 1 \Rightarrow p''(0) = 2$

$\Rightarrow 2c = 2$ or $c = 1$

$\therefore p(x) = ax^4 + bx^3 + x^2$

$\Rightarrow p'(x) = 4ax^3 + 3bx^2 + 2x$

As $p(x)$ has extremum at $x = 1$ and 2

$\therefore \quad p'(1) = 0$ and $p'(2) = 0$

$\Rightarrow \quad 4a + 3b + 2 = 0 \qquad$ ...(i)

$\Rightarrow \quad 32a + 12b + 4 = 0$ or $8a + 3b + 1 = 0 \qquad$ ...(ii)

Solving eq's (i) and (ii) we get $a = \frac{1}{4}$ and $b = -1$

$\therefore \quad p(x) = \frac{1}{4}x^4 - x^3 + x^2$

So, that $p(2) = \frac{16}{4} - 8 + 4 = 0$