PROBLEM

Let f: $\mathbb{R} \to (0, \infty)$ and g : $\mathbb{R} \to \mathbb{R}$ be twice differentiable functions such that f" and g" are continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If

$\lim_{x \to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then *(JEE Adv. 2016)*

(a) f has a local minimum at x=2

(b) f has a local maximum at x=2

(c) $f''(2) > f(2)$

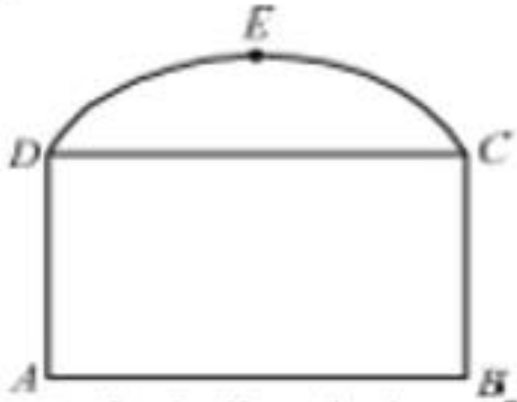
(d) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

SOLUTION

Let *ABCEDA* be the window as shown in the figure and let

$AB = x\ m$

$BC = y\ m$



Then its perimeter including the base *DC* of arch

$$= \left(2x + 2y + \frac{\pi x}{2}\right) m$$

$$\therefore \quad P = \left(2 + \frac{\pi}{2}\right) x + 2y \qquad ...(1)$$

Now, area of rectangle $ABCD = xy$

and area of arch $DCED = \frac{\pi}{2}\left(\frac{x}{2}\right)^2$

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Hence the area of light transmitted $= 3\lambda(xy) + \lambda\left[\frac{\pi}{2}\left(\frac{x}{2}\right)^2\right]$

$$\Rightarrow \quad A = \lambda\left[3xy + \frac{\pi x^2}{8}\right] \qquad \text{......(2)}$$

Substituting the value of y from (1) in (2), we get

$$A = \lambda\left[3x\frac{1}{2}\left[P - \left(\frac{4+\pi}{2}\right)x\right] + \frac{\pi x^2}{8}\right]$$

$$= \lambda\left[\frac{3Px}{2} - \frac{3(4+\pi)}{4}x^2 + \frac{\pi x^2}{8}\right]$$

$$\therefore \quad \frac{dA}{dx} = \lambda\left[\frac{3P}{2} - \frac{3(4+\pi)}{2}x + \frac{\pi x}{4}\right]$$

For A to be maximum $\frac{dA}{dx} = 01$

$$\Rightarrow \quad x = \frac{\frac{3P}{2}}{\frac{-\pi}{4} + \left(\frac{12+3\pi}{2}\right)}$$

$$\Rightarrow \quad x = \frac{3P}{2} \times \frac{4}{5\pi + 24} \Rightarrow x = \frac{6P}{5\pi + 24}$$

Also $\frac{d^2A}{dx^2} = \lambda\left[\frac{-3(4+\pi)}{2} + \frac{\pi}{4}\right] < 0$

$\therefore$ A is max when $x = \frac{6P}{5\pi + 24}$

[Using value of P from (1)]

$$\Rightarrow \quad (5\pi + 24 - 12 - 3\pi)x = 12y \Rightarrow (2\pi + 12)x = 12y$$

$$\Rightarrow \quad \frac{y}{x} = \frac{\pi + 6}{6}$$

$\therefore$ The required ratio of breadth to length of the rectangle $= 6 + \pi : 6$