31. If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

Sol. Let length of one edge of cube be x units and radius of sphere be r units.

Now surface area of cube $= 6x^2$

And surface area of sphere $= 4\pi r^2$

According to the question

$$6x^2 + 4\pi r^2 = k, \text{ where } k \text{ is constant}$$

$$\Rightarrow \quad x = \left[\frac{k - 4\pi r^2}{6}\right]^{1/2}$$

Now, volume of cube $= x^3$

and volume of sphere $= \frac{4}{3}\pi r^3$

The sum of volume of the cube and volume of the sphere is

$$S = x^3 + \frac{4}{3}\pi r^3$$

$$= \left[\frac{k - 4\pi r^2}{6}\right]^{3/2} + \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dS}{dr} = -2\pi r\left[\frac{k - 4\pi r^2}{6}\right]^{1/2} + 4\pi r^2 \qquad \text{(ii)}$$

$$= -2\pi r\left[\left\{\frac{k - 4\pi r^2}{6}\right\}^{1/2} - 2r\right]$$

Now, $\quad \frac{dS}{dr} = 0$

$$\Rightarrow \quad 2r = \left(\frac{k - 4\pi r^2}{6}\right)^{1/2} \qquad (\text{as } r \neq 0)$$

$$\Rightarrow \quad 4r^2 = \frac{k - 4\pi r^2}{6}$$

$$\Rightarrow \quad 24r^2 = k - 4\pi r^2$$

$$\Rightarrow \quad r^2[24 + 4\pi] = k$$

$$\Rightarrow \quad r = \sqrt{\frac{k}{24 + 4\pi}} = \frac{1}{2}\sqrt{\frac{k}{6 + \pi}}$$

Clearly, this is point of minima

For $r = \frac{1}{2}\sqrt{\frac{k}{6 + \pi}}$, $x = \left[\frac{k - 4\pi \cdot \frac{1}{4} \cdot \frac{k}{(6 + \pi)}}{6}\right]^{1/2}$

$$= \left[\frac{(6 + \pi)k - \pi k}{6(6 + \pi)}\right]^{1/2} = \left[\frac{k}{6 + \pi}\right]^{1/2} = 2r$$

Thus, the sum of their volume is minimum when $x = 2r$.

Hence, the ratio of an edge of cube to the diameter of the sphere is 1:1.