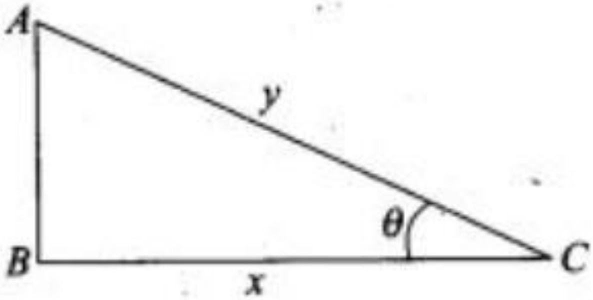
25. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\pi/3$.

Sol. Let ABC be a right angled triangle in which side $BC = x$ (say)

and hypotenuse $AC = y$ (say).

Given $x + y = k$ (const.) $\Rightarrow y = k - x$

Now the area of the triangle ABC is given by



$$A = \frac{1}{2} BC \cdot AB = \frac{1}{2} x\sqrt{(y^2 - x^2)} = \frac{1}{2} x\sqrt{[(k-x)^2 - x^2]}$$

Let $u = A^2 = \frac{1}{4} x^2 (k^2 - 2kx)$

$$\Rightarrow \qquad \frac{du}{dx} = \frac{1}{2} k(kx - 3x^2) \text{ and } \frac{d^2u}{dx^2} = \frac{1}{2} k(k - 6x)$$

For max. or min. of u, $\frac{du}{dx} = 0 \Rightarrow x = k/3$ $(\because x \neq 0)$

When $x = k/3$, $\frac{d^2u}{dx^2} = \frac{1}{2} k(k - 2k) = -\frac{1}{2} k^2 < 0$

$\Rightarrow u$, i.e. A, is max. when $x = k/3$ and when $y = k - x = 2k/3$

Now $\cos\theta = BC/AC = x/y = 1/2 \Rightarrow \theta = \pi/3$

Hence the required angle is $\pi/3$