Example 16 Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution Let ABCD be the rectangle of maximum area with sides AB = $2x$ and BC = $2y$, where C (x, y) is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as shown in the Fig.6.3.

The area A of the rectangle is $4xy$ i.e. $A = 4xy$ which gives $A^2 = 16x^2y^2 = s$ (say)

Therefore, $s = 16x^2\left(1-\frac{x^2}{a^2}\right).b^2 = \frac{16b^2}{a^2}\,(a^2x^2 - x^4)$

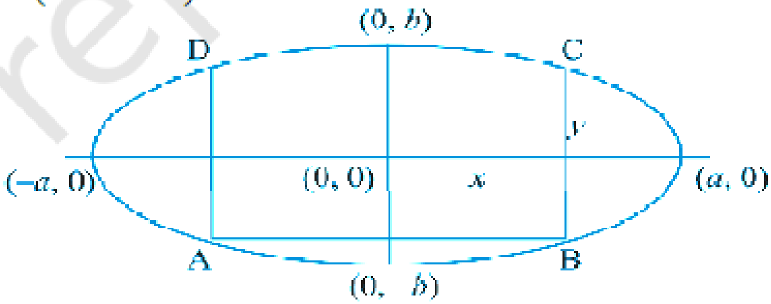

Fig. 6.3

$$\Rightarrow \quad \frac{ds}{dx} = \frac{16b^2}{a^2}\,.\,[2a^2x - 4x^3].$$

Again, $\frac{ds}{dx} = 0 \Rightarrow x = \frac{a}{\sqrt{2}}$ and $y = \frac{b}{\sqrt{2}}$

Now, $\frac{d^2s}{dx^2} = \frac{16b^2}{a^2}\,[2a^2 - 12x^2]$

At $x=\frac{a}{\sqrt{2}}$, $\frac{d^2s}{dx^2}=\frac{16b^2}{a^2}[2a^2-6a^2]=\frac{16b^2}{a^2}(-4a^2)<0$

Thus at $x = \frac{a}{\sqrt{2}}$, $y = \frac{b}{\sqrt{2}}$, s is maximum and hence the area A is maximum.

Maximum area $= 4.x.y = 4\,.\,\frac{a}{\sqrt{2}}\,.\,\frac{b}{\sqrt{2}} = 2ab$ sq units.