Example 15 Find the maximum and minimum values of

$$f(x) = \sec x + \log \cos^2 x,\ 0 < x < 2\pi$$

Solution $f(x) = \sec x + 2 \log \cos x$

Therefore, $f'(x) = \sec x \tan x - 2 \tan x = \tan x (\sec x - 2)$

$$f'(x) = 0 \Rightarrow \tan x = 0 \text{ or } \sec x = 2 \text{ or } \cos x = \frac{1}{2}$$

Therefore, possible values of x are $x = 0$, or $x = \pi$ and

$$x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3}$$

Again, $f''(x) = \sec^2 x (\sec x - 2) + \tan x (\sec x \tan x)$

$= \sec^3 x + \sec x \tan^2 x - 2\sec^2 x$

$= \sec x (\sec^2 x + \tan^2 x - 2\sec x)$. We note that

$f''(0) = 1 (1 + 0 - 2) = -1 < 0$. Therefore, $x = 0$ is a point of maxima.

$f''(\pi) = -1 (1 + 0 + 2) = -3 < 0$. Therefore, $x = \pi$ is a point of maxima.

$f''\left(\frac{\pi}{3}\right) = 2 (4 + 3 - 4) = 6 > 0$. Therefore, $x = \frac{\pi}{3}$ is a point of minima.

$f''\left(\frac{5\pi}{3}\right) = 2 (4 + 3 - 4) = 6 > 0$. Therefore, $x = \frac{5\pi}{3}$ is a point of minima.

Maximum Value of y at $x = 0$ is $1 + 0 = 1$

Maximum Value of y at $x = \pi$ is $-1 + 0 = -1$

Minimum Value of y at $x = \frac{\pi}{3}$ is $2 + 2 \log \frac{1}{2} = 2 (1 - \log 2)$

Minimum Value of y at $x = \frac{5\pi}{3}$ is $2 + 2 \log \frac{1}{2} = 2 (1 - \log 2)$