10. Show that $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$.

Sol. We have, $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$

$$\text{L.H.S.}=\cos\left(2\tan^{-1}\frac{1}{7}\right)$$

$$=\cos\left(\cos^{-1}\frac{1-\left(\frac{1}{7}\right)^2}{1+\left(\frac{1}{7}\right)^2}\right)\qquad\left(\because 2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}\right)$$

$$=\cos\left(\cos^{-1}\frac{48/49}{50/49}\right)$$

$$=\cos\left(\cos^{-1}\frac{24}{25}\right)=\frac{24}{25}\qquad(\because \cos(\cos^{-1}x)=x,\ x\in[-1,1])$$

$$\text{R.H.S.}=\sin\left(4\tan^{-1}\frac{1}{3}\right)$$

$$=\sin\left(2\left(2\tan^{-1}\frac{1}{3}\right)\right)$$

$$= \sin\left(2\left(\tan^{-1}\frac{2\cdot\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)\right) \quad \left(\because 2\tan^{-1}x \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)$$

$$= \sin\left(2\tan^{-1}\frac{2/3}{8/9}\right)$$

$$= \sin\left(2\tan^{-1}\frac{3}{4}\right)$$

$$= \sin\left(\sin^{-1}\frac{2\times\frac{3}{4}}{1+\left(\frac{3}{4}\right)^2}\right) \quad \left(\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right)$$

$$= \sin\left(\sin^{-1}\frac{3/2}{25/16}\right)$$

$$= \sin\left(\sin^{-1}\frac{24}{25}\right) = \frac{24}{25} \qquad (\because \sin(\sin^{-1}x) = x, x \in [-1, 1])$$

$\therefore$ L.H.S. = R.H.S.