Example 20 Show that

$$2\tan^{-1}\left\{\tan\frac{\alpha}{2}.\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}=\tan^{-1}\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}$$

Solution L.H.S. $=\tan^{-1}\dfrac{2\tan\frac{\alpha}{2}.\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$ $\left(\text{since } 2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}\right)$

$$=\tan^{-1}\frac{2\tan\frac{\alpha}{2}\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}}{1-\tan^2\frac{\alpha}{2}\left(\dfrac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)^2}$$

$$=\tan^{-1}\frac{2\tan\frac{\alpha}{2}.\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}\right)^2-\tan^2\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)^2}$$

$$= \tan^{-1} \frac{2\tan\frac{\alpha}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\beta}{2}\right)\left(1-\tan^2\frac{\alpha}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}$$

$$= \tan^{-1} \frac{\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \dfrac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\dfrac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}$$

$$= \tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right) = \text{R.H.S.}$$