Infinite Limits

Having discussed in detail limits as $x \to \pm\infty$, we would like to discuss in more detail limits where $f(x) \to \pm\infty$.

Once again we would like to emphasize that $\pm\infty$ are not numbers, so if we write

$$\lim_{x \to x_0} f(x) = \infty$$

we are not saying that the limit exists. What we are saying is that the limit does not exist, and it does not exist because the values of the function $f(x)$ grow without bound as $x \to x_0$.

Similarly, $\lim_{x \to x_0} f(x) = -\infty$ means that the limit as $x \to x_0$ of $f(x)$ does not exist because the function values decrease without bound (become arbitrarily large in magnitude, and negative in sign).

Throughout this discussion one-sided limits will be particularly useful to us, because it is quite common that a function may approach ∞ from one side of a point, and $-\infty$ from the other side of the point.

Definition: Infinite Limits

We write $\lim_{x \to x_0} f(x) = \infty$ if for every number $B > 0$ there exists a corresponding $\delta > 0$ such that for all x with $0 < |x - x_0| < \delta$ we have $f(x) > B$ Similarly, we write $\lim_{x \to x_0} f(x) = -\infty$ if for every number $-B < 0$ there exists a corresponding $\delta > 0$ such that for all x with $0 < |x - x_0| < \delta$ we have $f(x) < -B$

Arithmetic of infinite limits

The following are some informal rules for computing limits involving infinity. After I state them, I'll give a more formal version of each statement.

- $\frac{1}{0^+} = \infty$
- $\frac{1}{0^-} = -\infty$
- $\frac{1}{\infty} = 0$
- $\infty + \infty = \infty$
- $\infty \cdot \infty = \infty$

There are also several variants of these, such as $-\infty - \infty = -\infty$, and $(-\infty) \cdot (-\infty) = \infty$.

When I write the symbol $0+$, what I really mean is “the limit of positive numbers going to 0.”