* **Type - VIII** :→

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x + p\sin x\cos x + q}$$

where at least one of a, b, p, q is non-zero.

Method:
Multiply N^r & D^r by $\sec^2 x$, then put $\tan x = t$

* **Type - IX** :→

$$\int \frac{dx}{a\sin x + b\cos x + p}$$

where at least one of a, b, p is non-zero.

Method:-

Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ & $\cos x = \dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}$

then put $\tan\frac{x}{2} = t$.

* Type-10:-

$$\int \frac{l\sin x + m\cos x + n}{a\sin x + b\cos x + p}\,dx$$

where, at least one of a, b & p is non-zero

Method:-

Write:-

$$N^r = \lambda \cdot D^r + \mu \cdot \left\{\frac{d}{dx}(D^r)\right\} + \delta$$

& find λ, μ & δ by comparing the coefficients of $\sin x$, $\cos x$ & constants.

* Type-11:-

$$\int \sin^m x \cos^n x\,dx$$

Case (i):- If m & n, both are even whole no.:→

Convert into higher angles by:→

$\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$

Case (ii):- If at least one of m or n is odd:→

Let m → odd, put $\cos x = t$

n → odd, put $\sin x = t$

Case (iii):- If $m, n \in R$ & $(m+n)$ = negative integer:→

then put $\tan x = t$

* Type-12:→

$$\int (\sin x \pm \cos x) f\{\sin 2x\} dx$$

Put $\sin x \mp \cos x = t$

$1 \mp \sin 2x = t^2$

$$\boxed{\sin 2x = 1 \pm t^2}$$

Que. Evaluate :-

1. $\displaystyle\int \frac{dx}{x^2(1+x^7)^{6/7}} = \int \frac{dx}{x^8(x^{-7}+1)^{6/7}}$

$(x^{-7}+1) = t \Rightarrow \quad \frac{-7}{x^8}\,dx = dt$

$$= \int \frac{-\frac{dt}{7}}{t^{6/7}} = -\frac{1}{7}\cdot\frac{t^{-\frac{6}{7}+1}}{-\frac{6}{7}+1} + C$$

$$= \cancel{\frac{-1}{(x^{-7}+1)^{1/7}}} \quad - (t)^{1/7} + C$$

$$= -(x^{-7}+1)^{1/7} + C$$

$$= -\frac{(1+x^7)^{1/7}}{x} + C$$

2. $\displaystyle\int \sqrt[3]{x}\cdot\sqrt[7]{1+\sqrt[3]{x^4}}\,dx$

$$= \int x^{1/3}\cdot(x^{4/3}+1)^{1/7}\,dx$$

$x^{4/3}+1 = t \Rightarrow \quad \frac{4}{3}x^{1/3}\,dx = dt$

$$= \int \frac{3}{4}t^{1/7}\,dt = \frac{3}{4}\cdot\frac{t^{8/7}}{8/7} + C$$

$$= \frac{21}{32}(x^{4/3}+1)^{8/7} + C$$

$\frac{3}{2}$ $\int x^{2/5} (3 + x^{1/3})^2 \, dx$

Sol.

$= \int x^{2/5} (x^{2/3} + 9 + 2x^{1/3}) \, dx$

$= \int x^{16/15} \, dx + 9 \int x^{2/5} \, dx + 2 \int x^{11/15} \, dx$

$= \frac{15 x^{31/15}}{31} + \frac{45}{7} x^{7/5} + \frac{15}{13} x^{26/15} + C$

$\int x^{-1/2} (2 + 3x^{1/3})^{-2} \, dx$

Sol. $\int \frac{dx}{x^{1/2} (3x^{1/3} + 2)^2}$ $\quad x = t^6$, $dx = 6t^5 dt$

$= \int \frac{6t^5 \, dt}{t^3 (3t^2 + 2)^2} = \int \frac{6t^2}{(3t^2+2)^2} dt$

$= \cancel{2 \int \frac{dt}{3t^2 + 2} - 4 \int \frac{dt}{(3t^2+2)^2}}$

$= \cancel{\frac{2}{3 \times \sqrt{\frac{2}{3}}} \tan^{-1}\left(\frac{t + \sqrt{\frac{2}{3}}}{\sqrt{\frac{2}{3}}}\right) - 4 [}$

$= 6 \int \frac{t^2}{9t^4 + 4 + 12t^2} dt = 6 \int \frac{dt}{9t^2 + 12 + \frac{4}{t^2}}$

$$= \frac{1}{4}\left[\int \frac{\left(3 + \frac{2}{t^2}\right)}{\left(3t - \frac{2}{t}\right)^2 + 24}\,dt + \int \frac{3 - \frac{2}{t^2}}{\left(3t + \frac{2}{t}\right)^2}\,dt\right]$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{24}} \tan^{-1}\left(\frac{3t - \frac{2}{t}}{\sqrt{24}}\right) + - \frac{1}{\left(3t + \frac{2}{t}\right)^2}\right] + C$$

$$= \frac{1}{4}\left[\frac{1}{2\sqrt{6}} \tan^{-1}\left(\frac{3t^2 - 2}{t\sqrt{24}}\right) - \frac{t^2}{(3t^2 + 2)^2}\right] + C$$

$$= \frac{1}{4}\left[\frac{1}{2\sqrt{6}} \tan^{-1}\left(\frac{3x^{1/3} - 2}{x^{1/6}\sqrt{24}}\right) - \frac{x^{1/3}}{(3x^{1/3} + 2)^2}\right] + C \text{ Ans}$$

Que. Evaluate:-

(i) $$\int \frac{dx}{\sin^2 x + \sin 2x} = \int \frac{\sec^2 x}{(\tan^2 x + 2\tan x + 1) - 1}\,dx$$

$$= \int \frac{1}{(\tan x + 1)^2 - 1}\, d(\tan x)$$

$$= \frac{1}{2} \ln\left|\frac{\tan x + 1 - 1}{\tan x + 1 + 1}\right| + C$$

$$= \frac{1}{2} \ln\left|\frac{\tan x}{\tan x + 2}\right| + C \text{ Ans.}$$

(ii) $\displaystyle\int \frac{dx}{1+\sin x+\cos x} = \int \frac{dx}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2+2\cos^2\frac{x}{2}-1}$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{\cancel{\tan^2\frac{x}{2}}+\cancel{1}+2\tan\frac{x}{2}+\cancel{2}^{2}-\cancel{1}-\cancel{\tan^2\frac{x}{2}}}$$

$$= \frac{1}{2}\int \frac{d\left(\tan\frac{x}{2}\right)}{2\left(1+2\tan\frac{x}{2}\right)}$$

$$= \frac{1}{\cancel{4}} \ln\left|1+\tan\frac{x}{2}\right| \cdot \frac{1}{1/2} + C$$

$$= \frac{1}{\cancel{2}} \ln\left|1+\tan\frac{x}{2}\right| + C \quad \text{Ans.}$$

(iii) $\displaystyle\int \frac{5\cos x+6}{2\cos x+\sin x+3}\,dx$

Sol. $\displaystyle = \int \frac{2(2\cos x+\sin x+3)+(\cos x-2\sin x)}{2\cos x+\sin x+3}\,dx$ {Type 10}

$$= 2\int dx + \int \frac{\cos x - 2\sin x}{2\cos x+\sin x+3}\,dx$$

$$= 2x + \ln|2\cos x+\sin x+3| + C \quad \text{Ans.}$$

(iv) $\int \sin^5 x \cos^4 x \, dx$

Sol. $= -\int (1-\cos^2 x)^2 \cdot \cos^4 x \cdot (-\sin x) \, dx$

$\cos x = t \Rightarrow -\sin x \, dx = dt$

$= -\int (1-t^2)^2 \cdot t^4 \, dt$

$= -\left[\int (1+t^4-2t^2) \cdot t^4 \, dt\right.$

$= -\left[\frac{t^5}{5} - \frac{2}{7} \cdot t^7 + \frac{t^9}{9}\right] + C$

$= \left\{ \frac{2}{7}\cos^7 x - \frac{\cos^5 x}{5} - \frac{\cos^9 x}{9} \right. + C$ Ans.

Ques. Evaluate:-

(i) $\int \sec^4 x \, dx = \int \sec^2 x \, (1+\tan^2 x) \, dx$

$= \tan x + \frac{\tan^3 x}{3} + C$ Ans.

(ii) $\int (\sin x)^{1/3} \cdot (\cos x)^{-7/3} dx = \int \frac{(\tan x)^{1/3}}{\cos^2 x} dx$

$= \int (\tan x)^{1/3} \cdot d(\tan x) = \frac{3}{4} \tan^{4/3} x + C$

(iii) $\int \frac{\sin x + \cos x}{\sin^4 x + \cos^4 x} dx$

Sol. $= \int \frac{\sin x + \cos x \; dx}{1 - 2\sin^2 x \cos^2 x} = 2\int \frac{(\sin x + \cos x)}{2 - \sin^2 2x} dx$

$= 2\int \frac{\sin x + \cos x}{2 - \{(\sin x \mp \cos x)^2 \mp 1\}^2} dx$

$= 2\int \frac{d(\sin x \mp \cos x + 1)}{2 - \{\sin x - \cos x + 1\}^2} = 2\int \frac{dt}{2 - (t^2 \mp 1)^2}$

$= \frac{2}{2 \times \sqrt{2}} \ln \left| \frac{\sqrt{2} + \sin x - \cos x + 1}{\sqrt{2} - \sin x + \cos x - 1} \right| + C = 2\int \frac{dt}{-t^4 + 2t^2 + 1}$

$= \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2} + 1 + \sin x - \cos x}{\sqrt{2} - 1 - \sin x + \cos x} \right| + C$

$= -2\int \frac{dt}{t^4 - 2t^2 - 1} =$

(v) $\int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{2\sin x \cos x}}\, dx$$

$$= \sqrt{2}\int \frac{\sin x + \cos x}{\sqrt{1-(1-\sin 2x)}}\, dx = \sqrt{2}\int \frac{d(\sin x - \cos x)}{\sqrt{1-(\sin x - \cos x)^2}}$$

$$= \sqrt{2}\sin^{-1}(\sin x - \cos x) + C$$

(vi) $\int \frac{dx}{\sin x + \sec x} = 2\int \frac{\cos x\, dx}{2 + \sin 2x}$

$$= \int \frac{(\sin x + \cos x) + (\cos x - \sin x)}{2 + \sin 2x}\, dx$$

$$= \int \frac{\sin x + \cos x}{3 - (\sin x - \cos x)^2}\, dx + \int \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}\, dx$$

$$= \frac{1}{2\sqrt{3}} \ln\left| \frac{\sqrt{3} + \sin x - \cos x}{\sqrt{3} - \sin x + \cos x} \right|$$

$$+ \tan^{-1}(\sin x + \cos x) + C$$

dx

Reduction Formulae

Que. Let $I_n = \int \sec^n x \, dx$, $n \in I$

Sol. $\Rightarrow$

$$I_n = \int \underset{II}{\sec^2 x} . \underset{I}{\sec^{n-2} x} \, dx$$

$$= \sec^{n-2} x . \tan x - \int \tan x . (n-2) . \sec^{n-3} x . \sec x \tan x \, dx$$

$$= \sec^{n-2} x . \tan x - (n-2) \int (\sec^2 x - 1) . \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x . \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\Rightarrow \boxed{I_n = \frac{\sec^{n-2} x . \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}} , n \neq 1$$

$$\boxed{I_1 = \int \sec x \, dx = \ln |\sec x + \tan x|}$$

$$\therefore I_2 = \int \frac{\sec^0 x . \tan x}{2-1}$$

$$= \tan x$$

$$I_4 = \frac{\sec^2 x . \tan x}{3} + \frac{2}{3} \tan x$$

From formula:

$$(n-1) I_n - (n-2) I_{n-2} = \sec^{n-2}x \tan x \quad -①$$

$n=2$:

$I_2 = \tan x$

$n=3$: $2I_3 - I_1 = \tan x \sec x$

$n=4$: $3I_4 - 2I_2 = \tan x \cdot \sec^2 x$

Adding

$$-I_1 - I_2 + 2I_3 + 3I_4 = \tan x \,(1 + \sec x + \sec^2 x)$$

Que. 2 Let $I_n = \int \tan^n x \, dx, \quad n \in I$

Sol. $I_n = \int \tan^2 x \cdot \tan^{n-2}x \, dx = \int (\sec^2 x - 1) \tan^{n-2} x \, dx$

$$= \int \underset{II}{\sec^2 x} \cdot \underset{I}{\tan^{n-2}x} \, dx - \int \tan^{n-2}x \, dx$$

$$= \left[\tan^{n-2}x \cdot \tan x - \int \tan x \times (n-2)\tan^{(n-3)}x \cdot \sec^2 x \, dx\right] - I_{n-2}$$

$$= \left[\tan^{n-1}x - (n-2)\int \tan^{n-2}x \,(1 + \tan^2 x)\, dx\right] - I_{n-2}$$

$$I_n = \tan^{n-1}x - (n-2) I_{n-2} - (n-2)\underbrace{\int \tan^n x \, dx}_{I_n} - I_{n-2}$$

$$\boxed{I_n = \frac{\tan^{n-1}x}{n-1} - \frac{(n-3)}{n-1} I_{n-2}}$$

Ques 6 Evaluate :- $\int \cos^n x\,dx$

Sol

$$I_n = \int \underset{I}{\cos^{n-1}x} \cdot \underset{II}{\cos x}\,dx$$

$$= \sin x \cdot \cos^{n-1}x - \int \{ \sin x \cdot (n-1)\cos^{n-2}x \{-\sin x\}\,dx \}$$

$$I_n = \sin x \cos^{n-1}x - (n-1)\int \cos^{n-2}x + (n-1)\int \cos^n x\,dx$$

$$I_n = \sin x \cdot \cos^{n-1}x - (n-1) I_{n-2} + (n-1) I_n$$

$$\Rightarrow \boxed{I_n = \frac{\sin x \cdot \cos^{n-1}x}{n} - \frac{(n-1)}{n} I_{n-2}}$$