Integration of Various Types

Let Q be Quadratic Expression e.g. $Q = ax^2+bx+c$

& L is linear e.g. $\cancel{L = ax^2+bx+c}$

$L = lx+my+n$

Type-1 :→

$$\int \frac{1}{Q}\,dx, \quad \int \frac{1}{\sqrt{Q}}\,dx, \quad \int \sqrt{Q}\,dx$$

Method: Express 'Q' as completing the square & apply standard results.

Type-II:- $\int \frac{L}{Q}dx$, $\int \frac{L}{\sqrt{Q}}dx$, $\int L\sqrt{Q}\,dx$

Method: $L = \lambda \frac{d}{dx}(Q) + \mu$, then find 'λ' & 'μ' by comparing both sides

Type-III:- $\int \frac{1}{L_1\sqrt{L_2}}dx$, $\int \frac{1}{Q\sqrt{L_2}}dx$

Method:- Let $L_2 = t^2$

Type-IV:- $\int \frac{dx}{L\sqrt{Q}}$

Method:- Put $L = \frac{1}{t}$

Type-V:- $\int \frac{dx}{Q_1\sqrt{Q_2}}$

Method:- Put $x = \frac{1}{t}$

Type - VI :-

$$\int \frac{x^2 \pm a^2}{x^4 + kx^2 + a^4}\,dx,\quad \int \frac{x^2}{x^4 + kx^2 + a^4}\,dx,\quad \int \frac{1}{x^4 + kx^2 + a^4}\,dx$$

Method:- $\int \frac{x^2 + a^2}{x^4 + kx^2 + a^4}\,dx$

$$= \int \frac{1 + a^2/x^2}{x^2 + \frac{a^4}{x^2} + k}\,dx = \int \frac{1 + a^2/x^2}{(\underbrace{x - \frac{a^2}{x}}_{t})^2 + k + 2a^2}\,dx$$

$$\int \frac{dt}{t^2 + k + 2a^2}$$

Que: Evaluate:-

$= -\frac{1}{2} \tan^{-1}\{6(2t-1)\} + C$

$= -\frac{1}{2} \tan^{-1}\{6(\frac{2}{x+1} - 1)\}$

$= -\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$

$= -\frac{2}{\sqrt{3}} \tan^{-1}($

1. $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

Sol:

$x+1 = \frac{1}{t} \Rightarrow \quad dx = -\frac{1}{t^2} dt$

$= \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t}\sqrt{(1-\frac{1}{t})^2 + \frac{1}{t}}} = -\int \frac{dt}{\sqrt{t^2-t+1}}$

$= -\int \frac{dt}{\sqrt{(t-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}}$

$= -\ln\left|(t-\frac{1}{2}) + \sqrt{(t-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right| + C$

$= -\ln\left|(t-\frac{1}{2}) + \sqrt{t^2-t+1}\right| + C$ where

$\boxed{t = \frac{1}{x+1}}$

Ans

$$\int \frac{dx}{(2x^2+3)\sqrt{x^2-4}}$$

Sol.

$x = \frac{1}{t}$

$$= \int \frac{-\frac{1}{t^2}\,dt}{\frac{(2+3t^2)}{t^2}\frac{\sqrt{1-4t^2}}{t}} = -\int \frac{t\,dt}{(2+3t^2)\sqrt{1-4t^2}}$$

let $1-4t^2 = u^2$, $-8t\,dt = 2u\,du$

$$= \frac{1}{4}\int \frac{u\,du}{u\left\{2+3\left(\frac{1-u^2}{4}\right)\right\}} = \frac{1}{4}\int \frac{4\,du}{(8+3-3u^2)}$$

$$= \int \frac{du}{11-3u^2}$$

$$= \frac{1}{3}\int \frac{du}{\frac{11}{3}-u^2} = \frac{1}{3} \times \frac{1}{2\sqrt{\frac{11}{3}}} \ln\left|\frac{\sqrt{\frac{11}{3}}-u}{\sqrt{\frac{11}{3}}+u}\right| + C$$

$$= \frac{1}{2\sqrt{33}} \ln\left|\frac{\sqrt{11}-\sqrt{3}\,u}{\sqrt{11}+\sqrt{3}\,u}\right| + C$$

$$= \frac{1}{2\sqrt{33}} \ln\left|\frac{\sqrt{11}-\sqrt{3(1-4t^2)}}{\sqrt{11}+\sqrt{3(1-4t^2)}}\right| + C.$$

Ans.

$t = \frac{1}{x}$

Integration by parts

$$\int (\underbrace{u}_{1^{st}}\underbrace{v}_{2^{nd}})\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

The choice of first and Second funtion is chosen by ILATE rule.
The first function is chosen acording to below order:-

I → Inv. Trigo
L → log
A → Algebra
T → Trigo
E → exponential

Q) ~~Elab~~ Evaluate $\int x \tan^{-1}x \, dx$

$$\int \underbrace{x}_{II}\, \underbrace{\tan^{-1}x}_{I}$$

$$\Rightarrow \tan^{-1}x \cdot \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}$$

$$\tan^{-1}x \cdot \frac{x^2}{2} - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\,dx$$

$$\Rightarrow \tan^{-1}x \cdot x^2 - \frac{1}{2}\left[x - \tan^{-1}x\right] + C$$