Integration using Partial Fractions

Let $f(x)$ & $g(x)$ are polynomial f^n.

If degree of $f(x) < g(x)$, then $\frac{f(x)}{g(x)}$ is called proper fraction.

If degree of $f(x) \geq g(x)$ then $\frac{f(x)}{g(x)}$ is improper fraction

Partial fraction is applicable for proper fractions only.

If the denominator has linear / quadratic / their repeated ~~roots~~ factors, then partial fraction is applicable.

Case-1: If Dr has non-repeating linear factor only:→

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

Case-2. If Dr has non-repeating quadratic factors only:→

$$\frac{f(x)}{(x^2+4)(x^2-x+1)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2-x+1}$$

Case-3 If D^r has repeated linear / quadratic factors:-

$$\frac{f(x)}{(x-a)(x-b)^3(x-b)(x^2+2)(x^2-2x+5)^2} = \frac{A}{x-a} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2} + \frac{D}{(x-b)^3} + \frac{Ex+F}{x^2+2}$$

$$+ \frac{Gx+H}{x^2-2x+5} + \frac{Px+Q}{(x^2-2x+5)^2}$$

ved

Evaluate:-

fraction.

1. $\displaystyle\int \frac{x^2-4}{(x^2+1)(x^2+2)(x^2+3)}$

Sol. Let $x^2=y$

$$= \int \frac{y-4}{(y+1)(y+2)(y+3)} dx \quad \text{—— (1)}$$

$$\frac{y-4}{(y+1)(y+2)(y+3)} = \frac{A}{y+1} + \frac{B}{y+2} + \frac{C}{y+3}$$

$$y-4 = A(y+2)(y+3) + B(y+1)(y+3) + C(y+1)(y+2) \quad \text{—— (2)}$$

$y=-2$:

$$-6 = -B$$

$$\boxed{B=6}$$

$y=-3$

$$-7 = (-2)(-1) \times C$$

$$C = \frac{7}{2} \quad \text{—— (iii)}$$

$y=-1$

$$-5 = A(1)(2)$$

$$\boxed{A = -\frac{5}{2}}$$

$$\int \frac{y-4}{(y+1)(y+2)(y+3)}\,dy$$

$$= \int \left\{ \frac{-5}{2(y+1)} + \frac{6}{(y+2)} + \frac{(-11)}{2(y+3)} \right\} dx$$

$$= -\frac{5}{2}\int \frac{1}{x^2+1}\,dx + 6\int \frac{dx}{x^2+2} + \frac{-11}{2}\int \frac{dx}{x^2+3}$$

$$= -\frac{5}{2}\tan^{-1}(x) + 3\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{11}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

2. $$\int \frac{2x^3 - 3x^2 - 8x - 26}{2x^2 - 5x - 12}\,dx$$

Sol. $$= \int \left\{ x+1 + \frac{9x-4}{2x^2-5x-12} \right\} dx$$

$$= \int x\,dx + \int dx + \int \frac{9x-4}{2x^2-8x+3x-12}\,dx$$

$$= \frac{x^2}{2} + x + \int \frac{9x-4}{(x-4)(2x+3)}\,dx \quad \text{—— (1)}$$

$$\frac{9x-14}{(x-4)(2x+3)} = \frac{A}{x-4} + \frac{B}{(2x+3)}$$

$x=4$ $\quad 9\times 4-4 = A(2\times 4+3) \Rightarrow \boxed{A=2}$

$x=-\frac{3}{2}$ $\quad 9\times\left(-\frac{3}{2}\right)-14 = B\left(-\frac{3}{2}-4\right)$

$$\frac{-55}{2} = -\frac{11}{2}B \Rightarrow B=5$$

$\therefore$ $I = \frac{x^2}{2}+x+\int \frac{2}{x-4}\,dx+\int \frac{5}{2x-3}\,dx$

$= \frac{x^2}{2}+x+2\ln|x-4|+\frac{5}{2}\ln|2x-3|+C$ Ans.

3. $\int \frac{dx}{1+x^3}$

4. $\int \frac{dx}{\sin x(3+2\cos x)}$

5. $\int \frac{x^3+x-1}{(x^2+2)^2}\,dx$

Sol:

$$\int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)} \quad \text{—— (1)}$$

Let $\frac{1}{(x+1)(x^2+x+1)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+x+1}$

$= \frac{(A+B)x^2+x(A+C+B)+(A+C)}{(x+1)(x^2+x+1)}$

$\Rightarrow$ $A+B=0$, $\underbrace{A+B}_{0}+C=0$, $A+C=1$

$\Rightarrow \boxed{C=0} \Rightarrow \boxed{A=1} \Rightarrow \boxed{B=-1}$

$$\therefore \int \frac{dx}{x^3+1} = \int \frac{dx}{x+1} - \int \frac{x\,dx}{x^2+x+1}$$

$$= \int \ln|x+1| - \frac{1}{2}\int \frac{(2x+1)\,dx}{x^2+x+1} + \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$$= \ln|x+1| - \frac{1}{2}\ln|x^2+x+1| + \frac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \ln\left|\frac{x+1}{\sqrt{x^2+x+1}}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Sol 4: $= \int \frac{dx}{\sin x\,(3+2\cos x)}$

$$= \int \frac{\sin x\,dx}{(1-\cos^2 x)(3+2\cos x)} = -\int \frac{d(\cos x)}{(1-\cos^2 x)(3+2\cos x)}$$

$$= \int \frac{dt}{(t-1)(t+1)(2t+3)} \quad (1) \quad ; t = \cos x$$

Let $\frac{1}{(t-1)(t+1)(2t+3)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{2t+3}$

$$= \frac{A(t+1)(2t+3) + B(t-1)(2t+3) + C(t-1)(t+1)}{(t^2-1)(2t+3)}$$

$t=1$: $\quad 2A \times 5 = 1 \Rightarrow \boxed{A = 1/10}$

$t=-1$: $\quad 2B = 1 \Rightarrow \boxed{B = 1/2}$

$t = \frac{-3}{2}$ $\boxed{C = \frac{4}{5}}$

$\therefore = \int \frac{dt}{(t-1)(t+1)(2t+3)} = \frac{1}{10}\int \frac{dt}{t-1} + \frac{1}{2}\int \frac{dt}{t+1} + \frac{4}{5}\int \frac{dt}{2t+3}$

$= \frac{1}{10} \ln|t-1| + \frac{1}{2} \ln|t+1| + \frac{2}{5}\ln|2t+3| + C$

$= \frac{1}{10} \ln|\cos x - 1| + \frac{1}{2} \ln|\cos x + 1| + \frac{2}{5}\ln|2\cos x + 3| + C$

Q1.5 $\int \frac{x^3 + x - 1}{(x^2+2)^2} dx$ ①

Let $\frac{x^3+x-1}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$

~~$x^3+x-1 = (Ax+B)(x^4+4+2x^2) + (Cx+D)(x^2+D)$~~

$x^3+x-1) = (Ax+B)(x^2+2) + Cx + D$

$x^3+x-1 = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$

$x^3+x-1 = Ax^3 + x^2(B) + x(2A+C)x + (D+2B)$

$\boxed{A=1}$ $\boxed{B=0}$ $2A+C=1$ $\boxed{D=-1}$

$\boxed{C=-1}$