9. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

10. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$

11. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln \left| x + \sqrt{x^2-a^2} \right| + C$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln \left| x + \sqrt{x^2+a^2} \right| + C$

12. $\int \frac{dx}{|x| \sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C$

13. $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

14. $\int \sqrt{a^2+x^2}\, dx = \frac{x}{2} \sqrt{a^2+x^2} + \frac{a^2}{2} \ln \left| x + \sqrt{x^2+a^2} \right| + C$

$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2-a^2} \right| + C$

Algebra

Note: If $\int f(x)\,dx = g(x) + c$,

then $\int f(ax+b)\,dx = \dfrac{g(ax+b)}{a} + c$

All above formulae can be used as a result.

e.g. $\int 2^{4-5x}\,dx = \dfrac{2^{4-5x}}{(-5)\ln 2} + C$

Algebra of Integration:-

1. $\int \lambda f(x)\,dx = \lambda \int f(x)\,dx$

2. $\int \{\lambda f(x) \pm \mu g(x)\}\,dx \Leftrightarrow \lambda \int f(x)\,dx \pm \mu \int g(x)\,dx$

3. $\boxed{\int f(x)\cdot g(x)\,dx = f(x)\int g(x)\,dx - \int \left[\frac{d}{dx}\{f(x)\}\cdot \int g(x)\,dx\right]dx}$

By parts

4. **Substitution Method:-**

$\int g'(x)\cdot f\{g(x)\}\,dx = \int f(t)\,dt$, $t = g(x)$
$dt = g'(x)\,dx$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + C$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C$$

5. Cancellation of Integrals:-

$$\boxed{\int e^x \{f(x) + f'(x)\}\,dx = e^x f(x) + C}$$

$$\boxed{\int \{x f'(x) + f(x)\}\,dx = x f(x) + C}$$

Ques. Evaluate:-

(i) $\int \frac{(x-2)^3}{\sqrt{x}}\,dx$

(ii) $\int \frac{dx}{\sin^2 x \cos^2 x}$

(iii) $\int \frac{dx}{1+\sin x}$

(iv) $\int \sin^3 x \cos^3 x\,dx$

Sol (ii) $= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\,dx$

$= \int (\sec^2 x + \operatorname{cosec}^2 x)\,dx$

$= \tan x - \cot x + C$

(i) $\int x^3 - 6x^2 + 12x$

(iii) $\int \frac{1}{1+\sin x}\,dx = \int \frac{dx}{1+\cos\left(\frac{\pi}{2}-x\right)}$

$= \int \frac{dx}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)} = \frac{1}{2}\int \sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

$= \frac{1}{2} \times (-2)\tan\left(\frac{\pi}{4}-\frac{x}{2}\right) + C$

$= -\tan\left(\frac{\pi}{4}-\frac{x}{2}\right) + C$ Ans

(iv) $\int \sin^3 x \cos^3 x\, dx = \int \sin^3 x \cdot (1-\sin^2 x) \cdot \cos x\, dx$

$\sin x = t \qquad \cos x\, dx = dt$

$= \int (t^3 - t^5)\, dt$

$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C$ Ans

Alternate Method:-

$= \frac{1}{8}\int (2\sin x\cos x)^3 dx = \frac{1}{8}\int \sin^3 2x\, dx$

$= \frac{1}{8}\int \left(\frac{3\sin 2x - \sin 6x}{4}\right)dx = \frac{1}{32}\left[\frac{-3}{2}\cos 2x + \frac{1}{6}\cos 6x\right] + C$ Ans

$=$ ~~Ans~~

Alternate Method for (iii):-

$\int \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx = \int \frac{1-\sin x\, dx}{\cos^2 x}$

$= \int \sec^2 x\, dx - \int \tan x \sec x\, dx$

$= \tan x - \sec x + C$ Ans

~~Evaluate~~

Ques:- (v) $\int \sin x^\circ dx = \int \sin\left(\frac{\pi x}{180}\right)dx$

$= \frac{-180}{\pi}\cos\frac{\pi x}{180} + C = \frac{-180}{\pi}\cos x^\circ + C$

(vi) ~~$\int a^{mx} b^{nx} dx = \int e^{mx\ln a} \cdot e^{nx\ln b} dx = \int e^{x(m\ln a + n\ln b)} dx$~~

~~$= \int e^{(m+n)x}$~~

(vi) $\int a^{mx} \cdot b^{nx}\,dx = \int (a^m \cdot b^n)^x dx$

$$= \frac{(a^m \cdot b^n)^x}{\ln(a^m \cdot b^n)} + C \ \text{Ans}$$

(vii) $\int \frac{e^{3x} + e^{5x}}{e^x + e^{-x}}\,dx = \int \frac{e^x \cdot e^{3x}(e^{2x}+1)}{e^{2x}+1}\,dx$

$$= \frac{e^{4x}}{4} + C \ \text{Ans.}$$

Ques Evaluate:-

1. $\int \frac{x^2 + \cos^2 x}{(x^2+1)\sin^2 x}\,dx = \int \frac{x^2+1 - \sin^2 x}{(x^2+1)\sin^2 x}\,dx$

$$= \int \operatorname{cosec}^2 x\,dx - \int \frac{dx}{x^2+1}$$

$$= \cancel{\cot x} + \cancel{\tfrac{1}{2}\ln} \cancel{\tan^{-1} x + C\,\text{Ans}}$$

$$= -\cot x + \cot^{-1} x + C \ \text{Ans}$$

2. $\int \frac{dx}{(2x-7)\sqrt{(x-3)(x-4)}} = \int \frac{dx}{(2x-7)\sqrt{x^2-7x+12}}$

$$= \frac{1}{2}\int \frac{dx}{(x-\frac{7}{2})\sqrt{(x-\frac{7}{2})^2 - \frac{1}{2^2}}}$$

$$= \int \frac{2\,dx}{(2x-7)\sqrt{(2x-7)^2-1}}$$

Que. Evaluate :-

1. $\int \frac{xe^x}{(x+1)^2}\,dx = \int \left\{ \frac{(x+1)-1}{(x+1)^2} \right\} e^x\,dx$

$= \int e^x \cdot \underbrace{\frac{1}{x+1}}_{f(x)}\,dx + \int e^x \cdot \underbrace{\frac{(-1)}{(x+1)^2}}_{f'(x)}\,dx$ — (1)

$\therefore \int e^x \{f(x) + f'(x)\}$ form

$= \frac{e^x}{x+1} + C$ Ans.

2. $\int \frac{x+\sin x}{1+\cos x}\,dx = \int \cancel{\frac{x-\sin x + 2\sin x}{1+\cos x}}\,dx$

$= \int \cancel{\frac{x-\sin x}{1+\cos x}}\,dx + 2\int \cancel{\frac{\sin x}{1+\cos x}}\,dx$

$= \int \frac{x}{1+\cos x}\,dx + \int \frac{\sin x}{1+\cos x}\,dx$

$= \int \underset{\mathrm{I}}{2x} \underset{\mathrm{II}}{\sec^2} \frac{x}{2}\,dx + \int -\ln|1+\cos x| + C$

$= 2\left[2x \tan\frac{x}{2} - \int 2\tan\frac{x}{2}\,dx\right] - \ln|1+\cos x| + C$

$= 4x \tan\frac{x}{2} + 8\ln\left|\cos\frac{x}{2}\right| - \ln|1+\cos x| + C$ Ans.

3. $\int \frac{x\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$

Sol: Let $\sin^{-1}x = t \Rightarrow x = \sin t$

$\frac{1}{\sqrt{1-x^2}}\,dx = dt$

$= \int \frac{\sin t \, t}{1-\sin^2 t}\,.dt = \int \underset{\mathrm{I}}{t}\,(\underset{\mathrm{II}}{\sec t\tan t})\,dt$

$= t\sec t - \int \sec t\,dt$

$= t\sec t - \ln|\sec t + \tan t| + C$ Ans.

where $t = \sin^{-1}x$

4. $\int \left\{\frac{1}{\ln x} - \frac{1}{(\ln x)^2}\right\} dx$

Sol. $x = e^t \Rightarrow dx = e^t\,dt$

$= \int \left\{\frac{1}{t} - \frac{1}{t^2}\right\}.e^t\,dt = \frac{e^t}{t} + C$

$= \frac{x}{\ln x} + C$ Ans.