Example 1: Solve

$$(2x^3 + y^3)dx - 3xy^2 dy = 0$$

Solution: The given differential equation is a homogeneous differential equation of the first order since it has the form $M(x, y)dx + N(x, y)dy = 0$, where M(x,y) and N(x,y) are homogeneous functions of the same degree = 3 in this case.

Here, $M(x, y) = 2x^3 + y^3$ and $N(x, y) = -3xy^2$. To solve, we first rearrange the differential equation in the following format –

$$\frac{dy}{dx} = \frac{2x^3+y^3}{3xy^2} \quad \ldots\ldots (1)$$

To see that the equation is homogeneous, we can also see that the right-side can be converted to a function of the form $f(\frac{y}{x})$ as

$$\frac{2x^3}{3xy^2} + \frac{y^3}{3xy^2} = \frac{2}{3(\frac{y}{x})^2} + \frac{1}{3}(\frac{y}{x})$$

Scanned with CamScanner

Anyway, to solve now we proceed with the substitution $y = vx$ where x is the new dependent variable. Then the differential equation in the form 1) gets converted into

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{2x^3+y^3}{3xy^2} \\ &= \frac{2x^3+(vx)^3}{3x(vx)^2} \\ &= \frac{2+v^3}{3v^2} \end{aligned}$$

Then the differential equation can be converted to variables-separable form as

$$\begin{aligned} x\frac{dv}{dx} &= \frac{2+v^3}{3v^2} - v \\ &= \frac{2-2v^3}{3v^2} \\ \frac{3v^2}{v^3-1}\frac{dv}{dx} &= -\frac{2}{x} \end{aligned}$$

On integrating with respect to x, the general solution is found out to be

$$\begin{aligned} \int \frac{3v^2}{v^3-1}\frac{dv}{dx}dx &= \int -\frac{2}{x}dx \\ ln[mod(v^3-1)] &= -2lnx + c \end{aligned}$$

2021.06.11 17:00

Scanned with CamScanner

$$ln[mod(v^3 - 1)x^2] = c$$
$$[mod(v^3 - 1)]x^2 = e^c = C$$

This is the implicit form of the general solution for v(x). Now let us find y(x) by back-substituting $v = \frac{y}{x}$.

$$[mod((\frac{y}{x})^3 - 1)]x^2 = C$$
$$(\frac{y}{x})^3 - 1)x^2 = \pm C$$
$$y^3 - x^3 = \pm Cx$$

which is the final solution. It is actually a one-parameter family of curves which satisfies the homogeneous differential equation.

Scanned with CamScanner