Solve the differential equation

$$xy' = y\ln\frac{y}{x}.$$

Solution.

We notice that the root $x = 0$ does not belong to the domain of the differential equation. Rewrite the equation in the form:

$$y' = \frac{y}{x}\ln\frac{y}{x} = f\left(\frac{y}{x}\right).$$

As you can see, this equation is homogeneous.

Make the substitution $y = ux$. Hence,

$$y' = (ux)' = u'x + u.$$

Substituting this expression into the equation gives:

2021.06.11 17:00

$$x\left(u'x+u\right)=ux\ln\frac{ux}{x}.$$

Divide by $x\neq 0$ to get:

$$u'x+u=u\ln u,\ \Rightarrow\ \frac{du}{dx}x=u\ln u-u,\ \Rightarrow\ \frac{du}{dx}x=u\left(\ln u-1\right).$$

We obtain the separable equation:

$$\frac{du}{u\left(\ln u-1\right)}=\frac{dx}{x}.$$

The next step is to integrate the left and the right side of the equation:

$$\int\frac{du}{u\left(\ln u-1\right)}=\int\frac{dx}{x},\ \Rightarrow\int\frac{d\left(\ln u\right)}{\ln u-1}=\int\frac{dx}{x},\ \Rightarrow\int\frac{d\left(\ln u-1\right)}{\ln u-1}=\int\frac{dx}{x}.$$

Hence,

$$\ln\left|\ln u-1\right|=\ln\left|x\right|+C$$

$$\ln|\ln u - 1| = \ln|x| + C.$$

Here the constant C can be written as $\ln C_1$ $(C_1 > 0)$. Then

$$\ln|\ln u - 1| = \ln|x| + \ln C_1, \;\Rightarrow \ln|\ln u - 1| = \ln|C_1 x|, \;\Rightarrow \ln u - 1 = \pm C_1 x, \;\Rightarrow \ln u = 1 \pm C_1$$

Thus, we have got two solutions:

$$u = e^{1+C_1 x} \text{ and } u = e^{1-C_1 x}.$$

If $C_1 = 0$, the answer is $y = xe$ and we can make sure that it is also a solution to the equation. Indeed, substituting

$$y = xe, \;\; y' = e$$

into the differential equation, we obtain:

$$xe = xe \ln\frac{xe}{x}, \;\Rightarrow xe = xe\ln e, \;\Rightarrow xe = xe.$$

2021.06.11 17:00

Then all the solutions can be represented by one formula:

$$y = xe^{1+Cx},$$

where C is an arbitrary real number.