Solve the differential equation

$$(2x + y)dx - xdy = 0.$$

Solution.

It is easy to see that the polynomials $P(x, y)$ and $Q(x, y)$, respectively, at dx and dy, are homogeneous functions of the first order. Therefore, the original differential equation is also homogeneous.

Suppose that $y = ux$, where u is a new function depending on x. Then

$$dy = d(ux) = udx + xdu.$$

Substituting this into the differential equation, we obtain

$$(2x + ux)dx - x(udx + xdu) = 0.$$

Hence,

$$2xdx + \cancel{yxdx} - \cancel{xudx} - x^2du = 0.$$

Dividing both sides by x yields:

$$xdu = 2dx \text{ or } du = 2\frac{dx}{x}.$$

When dividing by x, we could lose the solution $x = 0$. The direct substitution shows that $x = 0$ is indeed a solution of the given differential equation.

Integrate the latter expression to obtain:

$$\int du = 2\int \frac{dx}{x} \quad \text{or} \quad u = 2\ln|x| + C,$$

where C is a constant of integration.

Returning to the old variable y, we can write:

$$y = ux = x\,(2\ln|x| + C).$$

Thus, the equation has two solutions:

$$y = x\left(2\ln|x| + C\right), \quad x = 0.$$