Example 4: Solve $(x \cos(y/x))(y\,dx + x\,dy) = y \sin(y/x)(x\,dy - y\,dx)$?

Solution:

The given equation may be written as

$(x \cos(y/x) + y \sin(y/x))y - (y \sin(y/x) - x \cos(y/x))\, x \,.\, dy/dx = 0$

$\Rightarrow dy/dx = \{x \cos(y/x) + y \sin(y/x)\}y \,/\, \{y \sin(y/x) - x \cos(y/x)\}x$

$\Rightarrow dy/dx = \{\cos(y/x) + (y/x)\sin(y/x)\}(y/x) \,/\, \{(y/x)\sin(y/x) - \cos(y/x)\}$

[Dividing numerator and denominator by x^2], which is clearly homogeneous, being a function of (y/x).

Putting $y = vx$ and $dy/dx = (v + x\,dv/dx)$ in it, we get v + x dv/dx = v(cos v + sin v)/(v sin v − cos v)

=> x dv/dx = [v(cos v + sin v)/(v sin v − cos v) -v]

=> x dv/dx = 2vcos v/(v sin v − cos v)

=>∫{(v sin v − cos v)/2vcos v}dv = ∫x dx [Integrating both sides]

=>∫tan v dv − ∫ dv/v = ∫ 2/x dx

=> -log | cos v | − log | v | + log C = 2 log | x |

=> log | cos v | + log | v | + 2log | x | = log | C |

=> log | x^2v cos v | = log | C |

=> | x^2v cos v | = C [After cancelling log on the both sides]

$\Rightarrow x^2 v \cos v = \pm C$

$\Rightarrow x^2 v \cos v = C_1$ [here we taking $\pm C = C_1$]

$\Rightarrow xy \cos(y/x) = C_1$, which is the required solution after putting the actual value of $v = y/x$