Example 3: Solve x dy/dx − y = $\sqrt{(x^2 + y^2)}$?

Solution:

The given equation may be written as dy/dx = {y + $\sqrt{(x^2 + y^2)}$}/x, which is clearly homogeneous.

Putting y = vx and dy/dx = v + x dv/dx in it, we get

v + x dv/dx = {vx + $\sqrt{(x^2 + v^2x^2)}$}/x

=> v + x dv/dx = v + $\sqrt{(1+v^2)}$ [After dividing the {vx + $\sqrt{(x^2 + v^2x^2)}$}/x]

=> x dv/dx = $\sqrt{(1 + v^2)}$ [v on the both sides gets cancelled]

=> dv/$\sqrt{(1+v^2)}$ = 1/x dx [after rearranging]

$\Rightarrow \int dv/\sqrt{(1+v^2)} = \int 1/x\, dx$ [integrating both sides]

$\Rightarrow \log | v | + \sqrt{(1+v^2)} | = \log | x | + \log C$

$\Rightarrow \log | \{v + \sqrt{(1+v^2)}\}/x | = \log | C |$

$\Rightarrow \{v + \sqrt{(1+v^2)}\}/x = \pm C$

$\Rightarrow v + \sqrt{(1+v^2)} = C_1 x$, where $C_1 = \pm C$

$\Rightarrow y + \sqrt{(x^2+y^2)} = C_1 x^2$, which is the required solution after putting the value of $v = y/x$