Example 2: Solve $(x\sqrt{(x^2 + y^2)} - y^2)dx + xy\, dy = 0$?

Solution:

The given equation may be written as

$dy/dx = y^2 - x\sqrt{(x^2 + y^2)}/xy$, which is clearly homogeneous

Putting $y = vx$ and $dy/dx = v + x\, dv/dx$ in it, we get

$v + x\, dv/dx = \{v^2x^2 - x\sqrt{(x^2 + v^2y^2)}\}/vx^2$

$=> x\, dv/dx = [\{v^2 - \sqrt{(1 + v^2)}\}/v - v]$

$=> x\, dv/dx = -\sqrt{(1 + v^2)}/v$

$=> \int v/\sqrt{(1 + v^2)}\, dv = -\int dx/xc$ [Integrating both the sides]

$$\Rightarrow \sqrt{(1+v^2)} = -\log|x| + C$$

$\Rightarrow \sqrt{(x^2+y^2)} + x \log|x| = Cx$, which is the required solution after putting the value of $v = y/x$.