If the marginal cost of producing x shoes is given by $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$ and the total cost of producing a pair of shoes is given by ₹12. Then find the total cost function.

Solution:

Given marginal cost function is $(x^2 + xy)\,dy + (3xy + y^2)dx = 0$

$$\frac{dy}{dx} = \frac{-(3xy + y^2)}{x^2 + xy} \qquad (1)$$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$$v + x\frac{dv}{dx} = \frac{-(3x\,vx + v^2x^2)}{x^2 + x\,vx}$$

$$= \frac{-(3v + v^2)}{1 + v}$$

Now,

$$x\frac{dv}{dx} = \frac{-3v - v^2}{1 + v} - v$$

$$= \frac{-3v - v^2 - v - v^2}{1 + v}$$

$$x\frac{dv}{dx} = \frac{-4v - 2v^2}{1 + v}$$

$$\frac{1+v}{4v+2v^2}\,dv = \frac{-dx}{x}$$

On Integration

$$\int \frac{1+v}{4v+2v^2}\,dv = -\int \frac{dx}{x}$$

Now, multiply 4 on both sides

$$\int \frac{4+4v}{4v+2v^2}\,dv = -4\int \frac{dx}{x}$$

$$\log (4v+2v^2) = -4 \log x + \log c$$

$$4v+2v^2 = \frac{c}{x^4}$$

$$x^4 (4v+2v^2) = c$$

Replace $v = \frac{y}{x}$

$$x^4\left(4\frac{y}{x}+2\frac{y^2}{x^2}\right) = c$$

$$x^4\left[\frac{4xy+2y^2}{x^2}\right] = c$$

$$c = 2x^2(2xy+y^2) \qquad (2)$$

Cost of producing a pair of shoes = ₹12

(*i.e*) $y = 12$ when $x = 2$

$$c = 8\left[48 + 144\right] = 1536$$

$\therefore$ The cost function is $x^2\,(2xy + y^2) = 768$