Find the particular solution of the differential equation $x^2\,dy + y\,(x + y)\,dx = 0$ given that $x = 1, y = 1$

Solution:

$$x^2dy + y(x+y)dx = 0$$

$$x^2dy = -y\,(x+y)\,dx$$

$$\frac{dy}{dx} = \frac{-(xy+y^2)}{x^2} \qquad (1)$$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$$v + x\frac{dv}{dx} = \frac{-(x\,vx + v^2x^2)}{x^2}$$

$$= -\left(v+v^2\right)$$

$$x\frac{dv}{dx} = -v^2 - v - v$$

$$= -(v^2+2v)$$

On separating the variables

$$\frac{dv}{v^2+2v} = \frac{-dx}{x}$$

$$\frac{dv}{v(v+2)} = \frac{-dx}{x}$$

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$$\frac{1}{2}\left[\frac{(v+2)-v}{v(v+2)}\right]dv = \frac{-dx}{x}$$

$$\frac{1}{2}\int\left(\frac{1}{v}-\frac{1}{v+2}\right)dv = -\int\frac{dx}{x}$$

$$\frac{1}{2}\left[\log v - \log(v+2)\right] = -\log x + \log c$$

$$\frac{1}{2}\log\frac{v}{v+2} = \log\frac{c}{x}$$

We have

$$\frac{v}{v+2} = \frac{c^2}{x^2}$$

Replace $v = \frac{y}{x}$, we get

$$\frac{y}{x\left(\frac{y}{x}+2\right)} = \frac{k}{x^2} \qquad \text{where } c^2 = k$$

$$\frac{y\,x^2}{y+2x} = \text{k} \qquad (2)$$

When $x = 1\,, y = 1$

$\therefore$ (2) $\Rightarrow$ $\quad k = \frac{1}{1+2}$

$1+2$

$$k = \frac{1}{3}$$

$\therefore$ The solution is $3x^2 y = 2x + y$