Solve the differential equation $y^2\,dx + (xy + x^2)dy = 0$

Solution

$$y^2dx + (xy + x^2)\,dy = 0$$

$$(xy + x^2)\,dy = -y^2dx$$

$$\frac{dy}{dx} = \frac{-y^2}{xy + x^2} \qquad (1)$$

It is a homogeneous differential equation, same degree in x and y

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x\,vx + x^2}$$

$$= \frac{-v^2}{v+1}$$

$$x\frac{dv}{dx} = \frac{-v^2}{v+1} - v$$

$$= \frac{-v^2 - v^2 - v}{v+1}$$

Scanned with CamScanner

$$x\frac{dv}{dx} = \frac{-\left(v+2v^2\right)}{1+v}$$

Now, separating the variables

$$\frac{1+v}{v(1+2v)}dv = \frac{-dx}{x}$$

$$\frac{(1+2v)-v}{v(1+2v)}dv = \frac{-dx}{x} \quad (\because\ 1+v = 1+2v-v)$$

$$\frac{1}{v}-\frac{1}{1+2v}dv = \frac{-dx}{x}$$

On Integration we have

$$\int\left(\frac{1}{v}-\frac{1}{1+2v}\right)dv = -\int\frac{dx}{x}$$

$$\log v - \frac{1}{2}\log(1+2v) = -\log x + \log c$$

$$\log\left(\frac{v}{\sqrt{1+2v}}\right) = \log\left(\frac{c}{x}\right)$$

$$\frac{v}{\sqrt{1+2v}} = \frac{c}{x}$$

Replace $v = \frac{y}{x}$ we get

$$\frac{\frac{y}{x}}{\sqrt{1+\frac{2y}{x}}} = \frac{c}{x}$$

$$\frac{y\sqrt{x}}{\sqrt{x+2y}} = c$$

$$\frac{y^2 x}{x+2y} = k$$

where $k = c^2$