Example: Solve $\frac{dy}{dx} = \frac{x-y}{x+y}$

Can we get it in $F(\frac{y}{x})$ style?

Start with: $\frac{x-y}{x+y}$

Divide through by x: $\frac{x/x-y/x}{x/x+y/x}$

Simplify: $\frac{1-y/x}{1+y/x}$

Yes! So let's go:

Start with: $\frac{dy}{dx} = \frac{1-y/x}{1+y/x}$

y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ $\quad v + x\frac{dv}{dx} = \frac{1-v}{1+v}$

Subtract v from both sides: $x\frac{dv}{dx} = \frac{1-v}{1+v} - v$

Then: $x\frac{dv}{dx} = \frac{1-v}{1+v} - \frac{v+v^2}{1+v}$

Simplify: $x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$

Now use Separation of Variables :

Separate the variables: $\frac{1+v}{1-2v-v^2}\,dv = \frac{1}{x}\,dx$

Put the integral sign in front: $\int\frac{1+v}{1-2v-v^2}\,dv = \int\frac{1}{x}\,dx$

Integrate: $-\frac{1}{2}\ln(1-2v-v^2) = \ln(x) + C$

Then we make **C = ln(k)**: $-\frac{1}{2}\ln(1-2v-v^2) = \ln(x) + \ln(k)$

Combine ln: $(1-2v-v^2)^{-\frac{1}{2}} = kx$

Square and Reciprocal: $1-2v-v^2 = \frac{1}{k^2x^2}$

Now substitute back $v = \frac{y}{x}$

Substitute $v = \frac{y}{x}$: $1-2(\frac{y}{x})-(\frac{y}{x})^2 = \frac{1}{k^2x^2}$

Multiply through by $\mathbf{x^2}$: $x^2-2xy-y^2 = \frac{1}{k^2}$

We are nearly there ... it is nice to separate out y though!

We can try to factor $x^2-2xy-y^2$ but we must do some rearranging first:

Change signs: $y^2+2xy-x^2 = -\frac{1}{k^2}$

Replace $-\frac{1}{k^2}$ by c: $y^2+2xy-x^2 = c$

Add $2x^2$ to both sides: $y^2+2xy+x^2 = 2x^2+c$

Factor: $(y+x)^2 = 2x^2+c$

Square root: $y+x = \pm\sqrt{(2x^2+c)}$

Subtract x from both sides: $y = \pm\sqrt{(2x^2+c)} - x$

And we have the solution.

The positive portion looks like this: