$$Evaluate: \int \frac{1}{x^4+1}dx$$

$$I = \int \frac{1}{x^4+1}dx \quad = \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \frac{1}{2}\int \left(\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} - \frac{x^2+\frac{1}{x^2}\big) dx}{x^2}\right)$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$$

Putting $x-\frac{1}{x}=u$ in first integral and $x+\frac{1}{x}=v$ in second integral, we get

$$I \quad = \frac{1}{2}\int \frac{du}{u^2+(\sqrt{2})^2} - \frac{1}{2}\int \frac{dv}{v^2-(\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+x\sqrt{2}+1}\right| + C$$