

Evaluate $\int \frac{\sqrt{x^2+1}\{\log_e(x^2+1)-2\log x\}dx}{x^4}$.

$$\begin{aligned} I &= \int \frac{\sqrt{x^2+1} \cdot \log\left(\frac{x^2+1}{x^2}\right) dx}{x^4} \\ &= \int \frac{\sqrt{1+\frac{1}{x^2}} \cdot \log\left(1+\frac{1}{x^2}\right) dx}{x^3} \end{aligned}$$

Let $1 + \frac{1}{x^2} = t$

$$\therefore -\frac{2}{x^3} dx = dt$$

$$\therefore I = -\frac{1}{2} \int \sqrt{t} \log t dt$$

$$= -\frac{1}{2} \left[\log t \cdot \frac{t^{3/2}}{3/2} - \int \frac{1}{t} \cdot \frac{t^{3/2}}{3/2} dt \right]$$

$$= -\frac{1}{2} \left[\frac{2}{3} t^{3/2} \log_e t - \frac{2}{3} \frac{t^{3/2}}{3/2} \right] + c$$

$$= -\frac{1}{3} t^{3/2} \log_e t + \frac{2}{9} t^{3/2} + c, \text{ where } t = 1 + \frac{1}{x^2}$$