Evaluate $\int \frac{dx}{x^3\sqrt{x^2-1}}$

Answer: Write $I = \int \frac{xdx}{x^4\sqrt{x^2-1}}$ and put $x^2 - 1 = t^2$, so that

$$2xdx = 2tdt$$

$$\therefore\ I = \int \frac{t}{(t^2+1)^2 t} dt = \int \frac{dt}{(t^2+1)^2}$$

$$\tan^{-1} t = \int \frac{dt}{t^2+1} = \int 1 \cdot \frac{1}{t^2+1} dt$$

$$= \frac{t}{t^2+1} + \int t \frac{2t}{(t^2+1)^2} dt$$

$$= \frac{t}{t^2+1} + 2\int \frac{t^2+1-1}{(t^2+1)^2} dt$$

$$= \frac{t}{t^2+1} + 2\tan^{-1} t - 2I$$

$$\therefore\ I = \frac{1}{2}\frac{t}{t^2+1} + \frac{1}{2}\tan^{-1} t$$

$$= \frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x^2} + \tan^{-1} \sqrt{x^2-1}\right) + C$$