

LECTURE NOTESConditional Probability:

Let A and B be any two events with $P(B) > 0$.
Then the conditional probability of A given that event B has already occurred,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

($P(A|B)$ is read as probability of A given B .)

eg. Suppose a fair die is rolled. What is the probability that 1 occurs given that an odd number occurs?

$$A \rightarrow 1 \text{ occurs} \quad P(A) = \frac{1}{6}$$

$$B \rightarrow \text{odd number occurs} \quad P(B) = \frac{3}{6} = \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{6}$$

$$\text{So, } P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Multiplication rules:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{--- (i)}$$

From (i),

$$P(A \cap B) = P(B) P(A|B) \quad \text{--- (ii)}$$

Similarly, if $P(A) > 0$, we can define

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(A) P(B|A) \quad \dots (iii)$$

(ii) and (iii) are called multiplication rules.

• For n events;

Let A_1, A_2, \dots, A_n be events with $P(\bigcap_{i=1}^n A_i) > 0$

This means that probability of occurrence of all events simultaneously is greater than zero.

This will ensure that probability of any event will be more than zero, hence all conditional probabilities are well defined.

$$\text{Then } P(\bigcap_{i=1}^n A_i) = P(A_1) P(A_2 | A_1) P(A_3 | A_1, A_2) \dots P(A_n | \bigcap_{i=1}^{n-1} A_i)$$

This is called general multiplication rule.

Proof: We will prove the rule using principle of mathematical induction.

For $n=1$, $P(A_1) = P(A_1)$
the statement is true.

Assume true for $n = k$

Then for $n = k+1$

$$\begin{aligned}
 P\left(\bigwedge_{i=1}^{k+1} A_i\right) &= P\left((A_1 \cap A_2) \bigwedge_{j=3}^{k+1} A_j\right) \\
 &= P(A_1) P(A_2 | A_1) \dots \\
 &= P(A_1 \cap A_2) P\left(\bigwedge_{j=3}^{k+1} A_j \mid (A_1 \cap A_2)\right) \\
 &= P(A_1) P(A_2 | A_1) \dots
 \end{aligned}$$

The proof is not really important. You can skip it.

Let A be the event that a person gets cured from a disease. B be the event that the person gets some medical treatment.

$$\text{Let } P(B) = 0.9, \quad P(A|B) = 0.8$$

Find probability of a person getting medical treatment and being cured from a disease.

We have to find $P(A \cap B)$

Use multiplication rule,

$$\begin{aligned}
 P(A \cap B) &= P(B) P(A|B) \\
 &= 0.9 \times 0.8 \\
 &= 0.72
 \end{aligned}$$

Total probability theorem:

Let B_1, B_2, \dots, B_n be pairwise disjoint and exhaustive events such that $P(B_i) > 0 \forall i$.

Then for any event A ,

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + P(A|B_n)P(B_n)$$

Proof:

We have $S = \bigcup_{i=1}^n B_i$, where S is sample space.

We can write, $A = A \cap S$

$$= A \cap \left(\bigcup_{i=1}^n B_i \right)$$

$$= \bigcup_{i=1}^n (A \cap B_i)$$

Since B_1, B_2, \dots, B_n are pairwise disjoint, the events

$A \cap B_1, A \cap B_2, \dots, A \cap B_n$ will also be pairwise disjoint.

Then by the axiom of additivity, we get

$$\begin{aligned} P(A) &= P\left(\bigcup_{i=1}^n (A \cap B_i)\right) = \sum_{i=1}^n P(A \cap B_i) \\ &= \sum_{i=1}^n P(A|B_i)P(B_i) \end{aligned}$$

Ex. A fair coin is tossed once. If a head comes up, a fair die is tossed once, and if a tail comes up, a fair die is tossed twice. We want to find the probability that at least a six is observed.

Let $H \rightarrow$ head comes up

$T \rightarrow$ tail comes up

$A \rightarrow$ a six is observed.

$$P(A) = P(A|H)P(H) + P(A|T)P(T)$$

$$= \frac{1}{6} \cdot \frac{1}{2} + \frac{11}{36} \cdot \frac{1}{2}$$

die is tossed once \rightarrow die is tossed twice
 so $P(\text{getting a six}) = \frac{1}{6}$ so $P(\text{getting at least one six}) = \frac{11}{36}$

11 elements $\left\{ (6,1), (6,2), \dots, (6,6) \right.$
 $\left. (1,6), (2,6), \dots, (5,6) \right\}$

$$= \frac{17}{72}$$

Bayes Theorem:

Let B_1, B_2, \dots, B_n be any disjoint events which are exhaustive with $P(B_i) > 0$ & $\sum_{i=1}^n P(B_i) = 1$

Let A be any event with $P(A) > 0$.

$$\text{Then, } P(B_r|A) = \frac{P(A|B_r)P(B_r)}{\sum_{i=1}^n P(A|B_i)P(B_i)} \quad ; r=1, 2, \dots, n$$

Proof:
$$P(B_r | A) = \frac{P(B_r \cap A)}{P(A)}$$

$$= \frac{P(A | B_r) P(B_r)}{\sum_{i=1}^n P(A | B_i) P(B_i)}$$
 using multiplication rule
 using theorem of total probability

Eg. A computer manufacturer procures chips from 3 suppliers B_1, B_2, B_3 in proportion $\frac{2}{5}, \frac{3}{10}$ and $\frac{3}{10}$ respectively. It is known from experience that 1% of chips from B_1 are defective, 5% of from B_2 are defective and 10% from B_3 .

A chip is randomly selected from the collection of the manufacturer. It is found to be defective. What is the probability that it was supplied by B_1 ?

$A \rightarrow$ chip is defective

$$P(A) = \sum_{i=1}^3 P(A | B_i) P(B_i)$$

$B_i \rightarrow$ the chip is supplied by B_i

$$\text{So, } P(A) = \frac{1}{100} \times \frac{2}{5} + \frac{5}{100} \times \frac{3}{10} + \frac{10}{100} \times \frac{3}{10}$$

$$= 0.049$$

$$P(B_1 | A) = \frac{P(A | B_1) P(B_1)}{P(A)} = \frac{\frac{1}{100} \times \frac{2}{5}}{0.049}$$

$$= \frac{4}{49}$$

Independence of events:

If occurrence of B does not affect the probability of A, then $P(A|B) = P(A)$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A \cap B) = P(A) P(B)}$$

You will get the same result if occurrence of A does not affect probability of B.

We define events A and B to be independent if $P(A \cap B) = P(A) P(B)$

eg. Suppose two fair dice are tossed simultaneously.

Let A \rightarrow even number on the first die.

B \rightarrow even number on the second die.

Check if A and B are independent.

$$P(A) = \frac{18}{36} = \frac{1}{2}$$

$$P(B) = \frac{18}{36} = \frac{1}{2}$$

$$P(A \cap B) = \frac{9}{36} = \frac{1}{4}$$

$$P(A \cap B) = P(A) P(B)$$

\Rightarrow A & B are independent.

Mutual independence:

We say events A, B, C are mutually independent

$$\left. \begin{aligned} P(A \cap B) &= P(A) P(B) \\ P(B \cap C) &= P(B) P(C) \\ P(A \cap C) &= P(A) P(C) \end{aligned} \right\} \text{pairwise independence}$$

and $P(A \cap B \cap C) = P(A) P(B) P(C)$

eg. Consider the previous example.

Let $C \rightarrow$ sum is even.

$$P(C) = \frac{18}{36} = \frac{1}{2}$$

$$P(A \cap C) P(A \cap B) = \frac{9}{36} = \frac{1}{4}$$

$$P(B \cap C) = \frac{9}{36} = \frac{1}{4}$$

$\Rightarrow A, B, C$ are pairwise independent

$$P(A \cap B \cap C) = \frac{9}{36} = \frac{1}{4} \neq P(A) P(B) P(C)$$

So, A, B & C are not mutually independent.