

Evaluate $\int \frac{x - \sin x}{1 - \cos x} dx$.

$$\begin{aligned} I &= \int \frac{x - \sin x}{1 - \cos x} dx \\ &= \int \frac{x}{1 - \cos x} dx - \int \frac{\sin x}{1 - \cos x} dx \\ &= \int \frac{x}{2} \operatorname{cosec}^2 \frac{x}{2} dx - \int \frac{2 \sin x/2 \cos x/2}{2 \sin^2 x/2} dx \\ &= \frac{1}{2} \int x \operatorname{cosec}^2 \frac{x}{2} dx - \int \cot \frac{x}{2} dx \\ &= \frac{1}{2} \left\{ x \left(-2 \cot \frac{x}{2} \right) - \int 1 \left(-2 \cot \frac{x}{2} \right) dx \right\} \\ &= -x \cot \frac{x}{2} + \int \cot \frac{x}{2} dx - \int \cot \frac{x}{2} dx + C \\ &= -x \cot \frac{x}{2} + C \end{aligned}$$