Circle

* Circle: Locus of a point which moves in a plane, so that its distance from a point in the plane remains constant.

* Equations of circle in different forms:

1. Centre-radius form: If a is the radius & (h,k) is the centre, then

$$(x-h)^2+(y-k)^2 = a^2.$$

2. Parametric form: If the radius of circle whose centre is at $(0,0)$ makes an angle θ with the positive direction of x-axis, then

$$x = a\cos\theta, \quad y = a\sin\theta. \qquad (0 \leq \theta < 2\pi).$$

When centre is at (h,k) then

$$x = h + a\cos\theta, \quad y = k + a\sin\theta.$$

• Equ^n of the chord of the circle $x^2+y^2=a^2$ joining $(a\cos\alpha, a\sin\alpha)$ & $(a\cos\beta, a\sin\beta)$ is

$$x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right) = a\cos\left(\frac{\alpha-\beta}{2}\right).$$

3. General form: $x^2+y^2+2gx+2fy+c=0$, where co-ordinates of the centre are $(-g,-f)$ & radius $= \sqrt{g^2+f^2-c}$.

4. Diametric form: Equ^n of circle on the line segment joining (x_1,y_1) & (x_2,y_2) as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

or $x^2+y^2-x(x_1+x_2)-y(y_1+y_2)+x_1x_2+y_1y_2=0.$

* Conditions of a circle for the second degree general equation:

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle, if,

i) Coefficient of x^2 = Coefficient of y^2.

ii) Coefficient of xy = zero.

* For concentric circles,

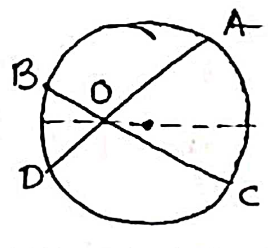
$$(x-h)^2 + (y-k)^2 = r_1.$$

$$(x-h)^2 + (y-k)^2 = r_2.$$

* Equation of circle passing through three non-collinear points: * ↓

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

* Cyclic Quadrilateral: If all four vertices of a quadrilateral lie on a circle, then the quadrilateral is called a cyclic quadrilateral. The four vertices are said to be concyclic.

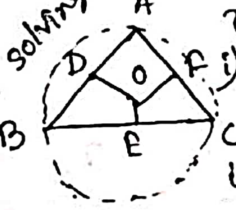


If A, B, C, D are concyclic, then

$$OA \cdot OD = OC \cdot OB$$

• Area of an equilateral triangle inscribed in the circle $x^2+y^2+2gx+2fy+c=0$ is $\frac{3\sqrt{3}}{4}(g^2+f^2-c)$ sq. units.

* ↑ Problem solving



$A \equiv (x_1, y_1)$ $B \equiv (x_2, y_2)$ $C \equiv (x_3, y_3)$ | D, E, F → mid-points

O → centre of circle

i) getting the point O by solving the eqn's of lines OD & OE.

ii) getting the radius OA

iii) forming the eqn of the circle.

* Intercepts made on the axes by a circle:

$$x^2+y^2+2gx+2fy+c=0.$$

x-intercept $= 2\sqrt{g^2-c}$.

y-intercept $= 2\sqrt{f^2-c}$.

* Position of a point with respect to a circle:

A point (x_1, y_1) lies outside, on or inside a circle $S \equiv x^2+y^2+2gx+2fy+c=0$ according as $S_1 > 0$, $=$, or < 0. $[S_1 \equiv x_1^2+y_1^2+2gx_1+2fy_1+c]$.

* Intersection of a line & a circle:

Equ^n of circle : $x^2+y^2=a^2$

Equ^n of line : $y = mx+c$.

Then, $(1+m^2)x^2+2mcx+c^2-a^2=0$.

$$x = \frac{-2mc \pm \sqrt{4m^2c^2-4(1+m^2)(c^2-a^2)}}{2(1+m^2)}.$$

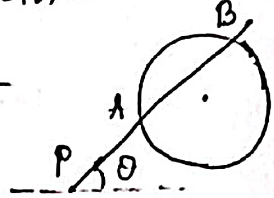
i) When points of intersection are real & distinct.

$4m^2c^2-4(1+m^2)(c^2-a^2) > 0 \Rightarrow a > \frac{|c|}{\sqrt{1+m^2}}$ ie the perpendicular from $(0,0)$ to $y=mx+c$.

ii) When points are coincident, $a = \frac{|c|}{\sqrt{1+m^2}}$.

iii) When points are imaginary, $a < \frac{|c|}{\sqrt{1+m^2}}$

* Product of the algebraical distances PA & PB is constant when from P, a secant be drawn to cut the circle in points A & B.



* The length of intercept cut off from the line $y = mx + c$, by the circle $x^2 + y^2 = a^2$ is

$$2\sqrt{\left\{\frac{a^2(1+m^2) - c^2}{(1+m^2)}\right\}}.$$

* Tangent to a circle:

1. Point form: Equ^n of tangent at the point (x_1, y_1) to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

2. Parametric form: Equ^n of tangent to the circle $x^2 + y^2 = a^2$ at the point $(a\cos\theta, a\sin\theta)$ is

$$x\cos\theta + y\sin\theta = a.$$

For $(x-a)^2 + (y-b)^2 = r^2$,

$$(x-a)\cos\theta + (y-b)\sin\theta = r$$

• Point of intersection of tangents at $(a\cos\theta, a\sin\theta)$ & $(a\cos\phi, a\sin\phi)$ is

$$\left(\frac{a\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \frac{a\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right)$$

3. Equ^n of a tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{1+m^2}$ & the co-ordinates of the point of contact are

$$\left(\pm\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right).$$

- The line $ax+by+c=0$ is tangent to the circle $x^2+y^2=a_1^2$ iff $c^2=a_1^2(a^2+b^2)$.
- If $y=mx+c$ is tangent to $x^2+y^2=r^2$ then point of contact $\left(-\frac{mr^2}{c}, \frac{r^2}{c}\right)$.
- If $ax+by+c=0$ is tangent to $x^2+y^2=r^2$, then point of contact $\left(-\frac{ar^2}{c}, -\frac{br^2}{c}\right)$.
- Condition that $lx+my+n=0$ touches $x^2+y^2+2gx+2fy+c=0$ is

$$(lg+mf-n)^2=(l^2+m^2)(g^2+f^2-c).$$

- Equ^n of tangent of $x^2+y^2+2gx+2fy+c=0$,

$$y+f=m(x+g)\pm\sqrt{(g^2+f^2-c)(1+m^2)}.$$

- Equ^n of tangent of slope m to the circle $(x-a)^2+(y-b)^2=r^2$ are given by

$$y-b=m(x-a)\pm r\sqrt{1+m^2}.$$

& the co-ordinates of points of contact

$$\left(a\pm\frac{mr}{\sqrt{1+m^2}}, b\mp\frac{r}{\sqrt{1+m^2}}\right).$$

* **Normal to a Circle :**

1. Point form: Equ^n of normal at point (x_1,y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is

$$\frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f}.$$

• Equⁿ of normal of $x^2+y^2=a^2$ at (x_1, y_1) is

$$\frac{x}{x_1} = \frac{y}{y_1}$$

2. Parametric form: Equⁿ of normal to the circle $x^2+y^2=a^2$ at the point $(a\cos\theta, a\sin\theta)$ is

$$y = x\tan\theta.$$

* Tangents from a point to the circle:

From a given point two tangents can be drawn to a circle which are real, co-incident or imaginary according as the given point lies outside, on or inside the circle.

* Length of the tangent from a point to circle:

The length of tangent from the point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is

$$\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c} = \sqrt{S_1}.$$

* Power of a point with respect to a circle:

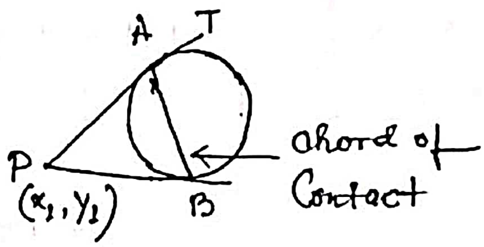
The power of a point $P(x_1, y_1)$ with respect to the circle $x^2+y^2+2gx+2fy+c=0$ is

$$x_1^2+y_1^2+2gx_1+2fy_1+c = S_1.$$

* Chord of Contact: Equⁿ of chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2+y^2=a^2$ is

$$xx_1+yy_1 = a^2.$$

$$\Rightarrow T = 0$$



- Equ^n of chord of contact at (x_1, y_1) with respect to circle $x^2+y^2+2gx+2fy+c=0$ is

$$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0 \Rightarrow T = 0.$$

* Chord bisected at a given point:

Equ^n of the chord of the circle $x^2+y^2=a^2$ bisected at the point (x_1, y_1) is given by

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 + (-a^2)$$

$$\Rightarrow \quad T = S_1.$$

- Equ^n of chord of circle $x^2+y^2+2gx+2fy+c=0$ that is bisected at (x_1, y_1) is $T = S_1$.

$$T = xx_1 + yy_1 + g(x+x_1) + (y+y_1)f + c$$

$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

* Pair of Tangents: Combined equ^n of the pair of tangents drawn from a point (x_1, y_1) to circle $x^2+y^2=a^2$ is

$$(x^2+y^2-a^2)(x_1^2+y_1^2-a^2) = (xx_1+yy_1-a^2)^2$$

$$\Rightarrow \quad SS_1 = T^2$$

- For the circle $x^2+y^2+2gx+2fy+c=0$.

$$(x^2+y^2+2gx+2fy+c)(x_1^2+y_1^2+2gx_1+2fy_1+c) =$$

$$\{xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c\}^2.$$

* Director circle: Locus of the point of intersection of two perpendicular tangents.

The equ^n of the director circle of circle $x^2+y^2=a^2$ is $x^2+y^2=2a^2$.

* Diameter of a circle: Locus of the middle points of a system of parallel chords of a circle.

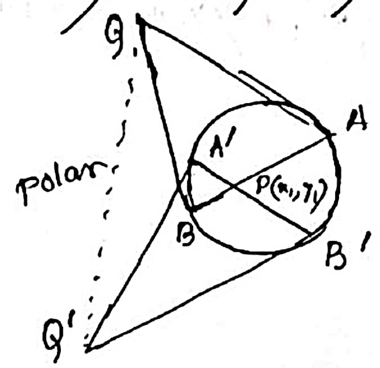
If the circle is $x^2+y^2=a^2$ & the equ^n of parallel chord is $y=mx+c$, locus of mid-point is

$$x+my=0.$$

* Pole & Polar: Let $P(x_1,y_1)$ be any point inside or outside the circle. Draw chords AB & A'B' passing through P. If tangents to the circle at A & B meet at $Q(h,k)$, then locus of Q is called the polar of P with respect to circle & P is called the pole. If tangents to the circle at A' & B' meet at Q' then the straight line QQ' is polar with p as its pole. If circle be $x^2+y^2=a^2$ then equ^n of polar $xx_1+yy_1=a^2$.

For the circle $x^2+y^2+2gx+2fy+c=0$.

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$



- Cor.

1. Co-ordinates of pole of a line:

pole of the line $lx+my+n=0$ with respect to the circle $x^2+y^2=a^2$:

$$\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right).$$

2. i) If the polar of $P(x_1, y_1)$ wrt a circle passes through $Q(x_2, y_2)$ then the polar of Q will pass through P & such point are <u>conjugate points</u>.

ii) If the pole of the line $ax+by+c=0$ wrt a circle lies on another line $a_1x+b_1y+c_1=0$, then the pole of the second line will lie on the first & such lines are said to be conjugate lines.

iii) The distance of any two points $P(x_1, y_1)$ & $Q(x_2, y_2)$ from the centre of a circle are proportional to the distance of each from the polar of the other.

iv) If O be the centre of a circle & P any point, then OP is perpendicular to the polar of P.

v) If O is centre & P any point, then if OP meet the polar of P in Q then $OP \cdot OQ = (\text{radius})^2$

* Two circles touching each other:

a) Touching externally: $|C_1C_2| = r_1 + r_2$
distance between their centres is sum of their radius.

b) Touching internally: $|C_1C_2| = |r_1 - r_2|$
distance between their centres is difference of their radii.

* Common tangents to two circles:

$$(x-x_1)^2 + (y-y_1)^2 = r_1^2$$
$$(x-x_2)^2 + (y-y_2)^2 = r_2^2.$$

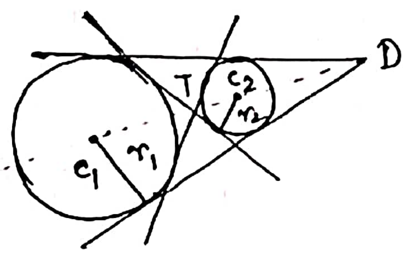
Different cases:

I. When $|C_1C_2| > r_1 + r_2$: Direct common tangent through (α, β): $y - \beta = m(x - \alpha)$. for m_1 & m_2, two common tangents.

Transverse common tangent through (γ, δ) is $y - \delta = M(x - \gamma)$.

$$D \equiv (\alpha, \beta) \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right).$$

$$T \equiv (\gamma, \delta) = \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right).$$



II. $|C_1C_2| = r_1 + r_2$: Direct common tangents:

$y - \beta = m(x - \alpha)$.

Transverse common tangent $S_1 - S_2 = 0$.

(same as equ^n of common chord).

III. $|C_1C_2| > |r_1 - r_2|$ & $|C_1C_2| < r_1 + r_2$.

direct common tangents are real & transverse common tangents are imaginary.

IV. $|C_1C_2| = |r_1 - r_2|$ equ^n of common tangent $S_1 - S_2 = 0$.

V. $|C_1C_2| < |r_1 - r_2|$ four common tangents are imaginary.

* Common chord of two circles:

equ^n of common chord of two circles $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ & $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ is

$$2x(g - g') + 2y(f - f') + c - c' = 0$$

$$\Rightarrow S - S' = 0.$$

- Length of common chord:

$$2\sqrt{(\text{rad. of circle } S)^2 - \left(\begin{matrix}\text{length of} \perp \text{from } C_1 \\ \text{on common chord}\end{matrix}\right)^2}$$

- Common chord becomes of max. length when it is a diameter of the smaller one between the circles.
- Circle on the common chord a diameter then centre of the circle passing through P & Q lie on the common chord of two circles.
- If the length of common chord is zero, then the two circles touch each other & the common chord becomes the common tangent.

* Family of circles:

1. Passing through the point of intersection of two given circles $S=0$, $S'=0$ is

$$S+\lambda S'=0. \quad (\lambda \neq -1)$$

2. Passing through the point of intersection of circle $S=0$ & a line $L=0$ is

$$S+\lambda L=0$$

3. Touching circle $S=0$ & the line $L=0$,

$$S+\lambda L=0.$$

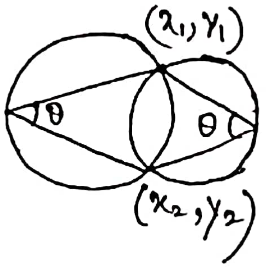
4. Passing through two given points (x_1, y_1) & (x_2, y_2)

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0.$$

5. Touch $y-y_1 = m(x-x_1)$ at (x_1, y_1) for any finite m is

$$(x-x_1)^2+(y-y_1)^2+\lambda\{(y-y_1)-m(x-x_1)\}=0.$$

If m is infinite the family of circles $(x-x_1)^2+(y-y_1)^2+\lambda(x-x_1)=0.$

6. 

Equ^n of the circles on diagram.

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) \pm \cot\theta\{(x-x_1)(y-y_2)-(x-x_2)(y-y_1)\}=0.$$