Example 10.1 The two thigh bones (femurs), each of cross-sectional area 10 cm² support the upper part of a human body of mass 40 kg. Estimate the average pressure sustained by the femurs.

Answer Total cross-sectional area of the femurs is $A = 2 \times 10\ \text{cm}^2 = 20 \times 10^{-4}\ \text{m}^2$. The force acting on them is F = 40 kg wt = 400 N (taking $g = 10\ \text{m s}^{-2}$). This force is acting vertically down and hence, normally on the femurs. Thus, the average pressure is

$$P_{av} = \frac{F}{A} = 2 \times 10^5\ \text{N m}^{-2}$$

10.2.1 Pascal's Law

The French scientist Blaise Pascal observed that the pressure in a fluid at rest is the same at all points if they are at the same height. This fact may be demonstrated in a simple way.

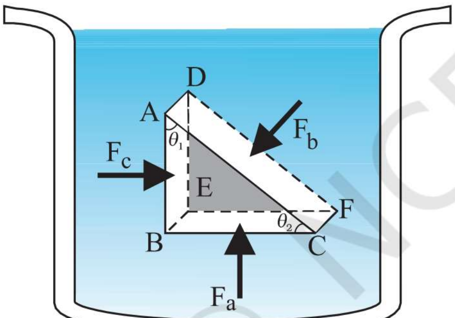


Fig. 10.2 *Proof of Pascal's law. ABC-DEF is an element of the interior of a fluid at rest. This element is in the form of a right-angled prism. The element is small so that the effect of gravity can be ignored, but it has been enlarged for the sake of clarity.*

Fig. 10.2 shows an element in the interior of a fluid at rest. This element ABC-DEF is in the form of a right-angled prism. In principle, this prismatic element is very small so that every part of it can be considered at the same depth from the liquid surface and therefore, the effect of the gravity is the same at all these points. But for clarity we have enlarged this element. The forces on this element are those exerted by the rest of the fluid and they must be normal to the surfaces of the element as discussed above. Thus, the fluid exerts pressures P_a, P_b and P_c on this element of area corresponding to the normal forces F_a, F_b and F_c as shown in Fig. 10.2 on the faces BEFC, ADFC and ADEB denoted by A_a, A_b and A_c respectively. Then

$F_b \sin\theta = F_c$, $\quad F_b \cos\theta = F_a$ (by equilibrium)

$A_b \sin\theta = A_c$, $\quad A_b \cos\theta = A_a$ (by geometry)

Thus,

$$\frac{F_b}{A_b} = \frac{F_c}{A_c} = \frac{F_a}{A_a}; \quad P_b = P_c = P_a \tag{10.4}$$

Hence, pressure exerted is same in all directions in a fluid at rest. It again reminds us that like other types of stress, pressure is not a vector quantity. No direction can be assigned to it. The force against any area within (or bounding) a fluid at rest and under pressure is normal to the area, regardless of the orientation of the area.

Now consider a fluid element in the form of a horizontal bar of uniform cross-section. The bar is in equilibrium. The horizontal forces exerted at its two ends must be balanced or the pressure at the two ends should be equal. This proves that for a liquid in equilibrium the pressure is same at all points in a horizontal plane. Suppose the pressure were not equal in different parts of the fluid, then there would be a flow as the fluid will have some net force acting on it. Hence in the absence of flow the pressure in the fluid must be same everywhere in a horizontal plane.

10.2.2 Variation of Pressure with Depth

Consider a fluid at rest in a container. In Fig. 10.3 point 1 is at height h above a point 2. The pressures at points 1 and 2 are P_1 and P_2 respectively. Consider a cylindrical element of fluid having area of base A and height h. As the fluid is at rest the resultant horizontal forces should be zero and the resultant vertical forces should balance the weight of the element. The forces acting in the vertical direction are due to the fluid pressure at the top (P_1A) acting downward, at the bottom (P_2A) acting upward. If mg is weight of the fluid in the cylinder we have

$$(P_2 - P_1)\ A = mg \tag{10.5}$$

Now, if ρ is the mass density of the fluid, we have the mass of fluid to be $m = \rho V = \rho h A$ so that

$$P_2 - P_1 = \rho g h \tag{10.6}$$

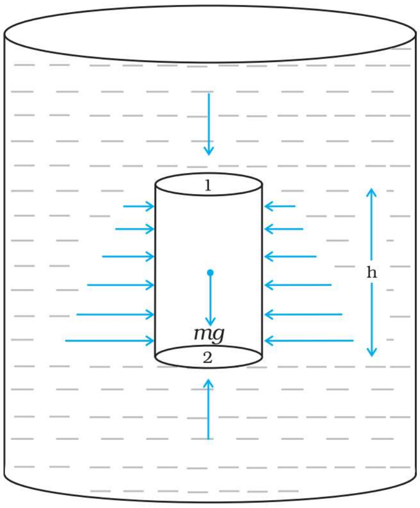


Fig.10.3 *Fluid under gravity. The effect of gravity is illustrated through pressure on a vertical cylindrical column.*

Pressure difference depends on the vertical distance h between the points (1 and 2), mass density of the fluid ρ and acceleration due to gravity g. If the point 1 under discussion is shifted to the top of the fluid (say, water), which is open to the atmosphere, P_1 may be replaced by atmospheric pressure (P_a) and we replace P_2 by P. Then Eq. (10.6) gives

$$P = P_a + \rho gh \quad (10.7)$$

Thus, the pressure P, at depth below the surface of a liquid open to the atmosphere is greater than atmospheric pressure by an amount ρgh. The excess of pressure, $P - P_a$, at depth h is called a **gauge pressure** at that point.

The area of the cylinder is not appearing in the expression of absolute pressure in Eq. (10.7). Thus, the height of the fluid column is important and not cross-sectional or base area or the shape of the container. The liquid pressure is the same at all points at the same horizontal level (same depth). The result is appreciated through the example of **hydrostatic paradox**. Consider three vessels A, B and C [Fig.10.4] of different shapes. They are connected at the bottom by a horizontal pipe. On filling with water, the level in the three vessels is the same, though they hold different amounts of water. This is so because water at the bottom has the same pressure below each section of the vessel.

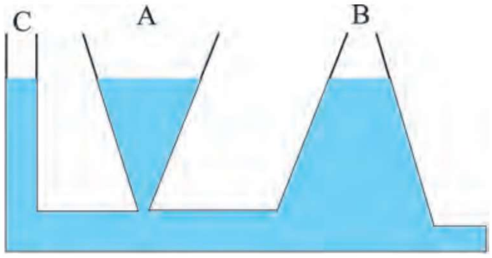


Fig 10.4 *Illustration of hydrostatic paradox. The three vessels A, B and C contain different amounts of liquids, all upto the same height.*

Example 10.2 What is the pressure on a swimmer 10 m below the surface of a lake?

Answer Here

h = 10 m and ρ = 1000 kg m^{-3}. Take g = 10 m s^{-2}

From Eq. (10.7)

$P = P_a + \rho gh$

$= 1.01 \times 10^5$ Pa + 1000 kg $m^{-3} \times 10$ m $s^{-2} \times 10$ m

$= 2.01 \times 10^5$ Pa

≈ 2 atm

This is a 100% increase in pressure from surface level. At a depth of 1 km, the increase in pressure is 100 atm! Submarines are designed to withstand such enormous pressures. ◀

10.2.3 Atmospheric Pressure and Gauge Pressure

The pressure of the atmosphere at any point is equal to the weight of a column of air of unit cross-sectional area extending from that point to the top of the atmosphere. At sea level, it is 1.013×10^5 Pa (1 atm). Italian scientist Evangelista Torricelli (1608–1647) devised for the first time a method for measuring atmospheric pressure. A long glass tube closed at one end and filled with mercury is inverted into a trough of mercury as shown in Fig.10.5 (a). This device is known as 'mercury barometer'. The space above the mercury column in the tube contains only mercury vapour whose pressure P is so small that it may be neglected. Thus, the pressure at Point A=0. The pressure inside the coloumn at Point B must be the same as the pressure at Point C, which is atmospheric pressure, P_a.

$$P_a = \rho gh \quad (10.8)$$

where ρ is the density of mercury and h is the height of the mercury column in the tube.

In the experiment it is found that the mercury column in the barometer has a height of about 76 cm at sea level equivalent to one atmosphere (1 atm). This can also be obtained using the value of ρ in Eq. (10.8). A common way of stating pressure is in terms of cm or mm of mercury (Hg). A pressure equivalent of 1 mm is called a torr (after Torricelli).

1 torr = 133 Pa.

The mm of Hg and torr are used in medicine and physiology. In meteorology, a common unit is the bar and millibar.

1 bar = 10^5 Pa

An open tube manometer is a useful instrument for measuring pressure differences. It consists of a U-tube containing a suitable liquid i.e., a low density liquid (such as oil) for measuring small pressure differences and a high density liquid (such as mercury) for large pressure differences. One end of the tube is open to the atmosphere and the other end is connected to the system whose pressure we want to measure [see Fig. 10.5 (b)]. The pressure P at A is equal to pressure at point B. What we normally measure is the gauge pressure, which is $P - P_a$, given by Eq. (10.8) and is proportional to manometer height h.

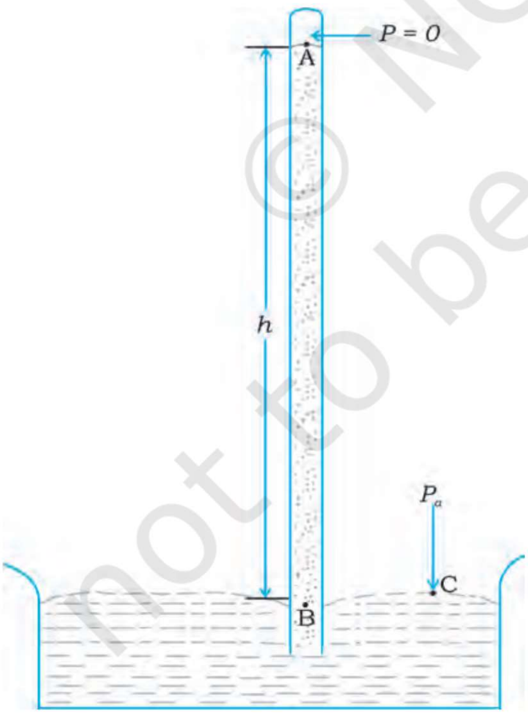
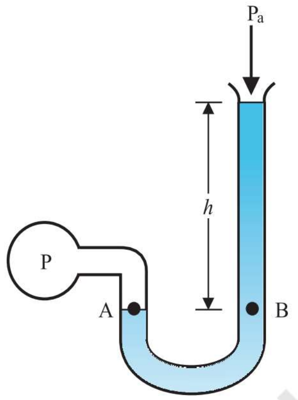


Fig 10.5 *(a) The mercury barometer.*



(b) The open tube manometer

Fig 10.5 *Two pressure measuring devices.*

Pressure is same at the same level on both sides of the U-tube containing a fluid. For liquids, the density varies very little over wide ranges in pressure and temperature and we can treat it safely as a constant for our present purposes. Gases on the other hand, exhibits large variations of densities with changes in pressure and temperature. Unlike gases, liquids are, therefore, largely treated as incompressible.

Example 10.3 The density of the atmosphere at sea level is 1.29 kg/m^3. Assume that it does not change with altitude. Then how high would the atmosphere extend?

Answer We use Eq. (10.7)

$\rho gh = 1.29 \text{ kg m}^{-3} \times 9.8 \text{ m s}^2 \times h \text{ m} = 1.01 \times 10^5 \text{ Pa}$

$\therefore h = 7989 \text{ m} \approx 8 \text{ km}$

In reality the density of air decreases with height. So does the value of g. The atmospheric cover extends with decreasing pressure over 100 km. We should also note that the sea level atmospheric pressure is not always 760 mm of Hg. A drop in the Hg level by 10 mm or more is a sign of an approaching storm.

Example 10.4 At a depth of 1000 m in an ocean (a) what is the absolute pressure? (b) What is the gauge pressure? (c) Find the force acting on the window of area 20 cm × 20 cm of a submarine at this depth, the interior of which is maintained at sea-level atmospheric pressure. (The density of sea water is 1.03 × 10^3 kg m^{-3}, g = 10 m s^{-2}.)

Answer Here $h = 1000$ m and $\rho = 1.03 \times 10^3$ kg m^{-3}.

(a) From Eq. (10.6), absolute pressure

$P = P_a + \rho g h$

$= 1.01 \times 10^5$ Pa

$+ 1.03 \times 10^3 \text{ kg m}^{-3} \times 10 \text{ m s}^{-2} \times 1000 \text{ m}$

$= 104.01 \times 10^5$ Pa

≈ 104 atm

(b) Gauge pressure is $P - P_a = \rho g h = P_g$

$P_g = 1.03 \times 10^3 \text{ kg m}^{-3} \times 10 \text{ ms}^2 \times 1000 \text{ m}$

$= 103 \times 10^5$ Pa

≈ 103 atm

(c) The pressure outside the submarine is $P = P_a + \rho g h$ and the pressure inside it is P_a. Hence, the net pressure acting on the window is gauge pressure, $P_g = \rho g h$. Since the area of the window is $A = 0.04$ m^2, the force acting on it is

$F = P_g A = 103 \times 10^5 \text{Pa} \times 0.04 \text{m}^2 = 4.12 \times 10^5$ N ◀

10.2.4 Hydraulic Machines

Let us now consider what happens when we change the pressure on a fluid contained in a vessel. Consider a horizontal cylinder with a piston and three vertical tubes at different points [Fig. 10.6 (a)]. The pressure in the horizontal cylinder is indicated by the height of liquid column in the vertical tubes. It is necessarily the same in all. If we push the piston, the fluid level rises in all the tubes, again reaching the same level in each one of them.

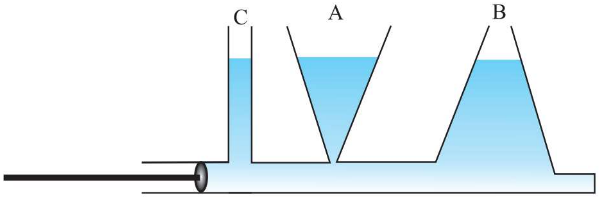


Fig 10.6 *(a) Whenever external pressure is applied on any part of a fluid in a vessel, it is equally transmitted in all directions.*

This indicates that when the pressure on the cylinder was increased, it was distributed uniformly throughout. We can say **whenever external pressure is applied on any part of a fluid contained in a vessel, it is transmitted undiminished and equally in all directions. This is another form of the Pascal's law and it has many applications in daily life.**

A number of devices, such as **hydraulic lift** and **hydraulic brakes,** are based on the Pascal's law. In these devices, fluids are used for transmitting pressure. In a hydraulic lift, as shown in Fig. 10.6 (b), two pistons are separated by the space filled with a liquid. A piston of small cross-section A_1 is used to exert a force F_1 directly on the liquid. The pressure $P = \dfrac{F_1}{A_1}$ is transmitted throughout the liquid to the larger cylinder attached with a larger piston of area A_2, which results in an upward force of $P \times A_2$. Therefore, the piston is capable of supporting a large force (large weight of, say a car, or a truck,

Archemedes' Principle

Fluid appears to provide partial support to the objects placed in it. When a body is wholly or partially immersed in a fluid at rest, the fluid exerts pressure on the surface of the body in contact with the fluid. The pressure is greater on lower surfaces of the body than on the upper surfaces as pressure in a fluid increases with depth. The resultant of all the forces is an upward force called buoyant force. Suppose that a cylindrical body is immersed in the fluid. The upward force on the bottom of the body is more than the downward force on its top. The fluid exerts a resultant upward force or buoyant force on the body equal to $(P_2 - P_1) \times A$ (Fig. 10.3). We have seen in equation 10.4 that $(P_2 - P_1)A = \rho g h A$. Now, hA is the volume of the solid and $\rho h A$ is the weight of an equivaliant volume of the fluid. $(P_2 - P_1)A = mg$. Thus, the upward force exerted is equal to the weight of the displaced fluid.

The result holds true irrespective of the shape of the object and here cylindrical object is considered only for convenience. This is Archimedes' principle. For totally immersed objects the volume of the fluid displaced by the object is equal to its own volume. If the density of the immersed object is more than that of the fluid, the object will sink as the weight of the body is more than the upward thrust. If the density of the object is less than that of the fluid, it floats in the fluid partially submerged. To calculate the volume submerged, suppose the total volume of the object is V_s and a part V_p of it is submerged in the fluid. Then, the upward force which is the weight of the displaced fluid is $\rho_f g V_p$, which must equal the weight of the body; $\rho_s g V_s = \rho_f g V_p$ or $\rho_s/\rho_f = V_p/V_s$ The apparent weight of the floating body is zero.

This principle can be summarised as; 'the loss of weight of a body submerged (partially or fully) in a fluid is equal to the weight of the fluid displaced'.

placed on the platform) $F_2 = PA_2 = \frac{F_1 A_2}{A_1}$. By changing the force at A_1, the platform can be moved up or down. Thus, the applied force has been increased by a factor of $\frac{A_2}{A_1}$ and this factor is the mechanical advantage of the device. The example below clarifies it.

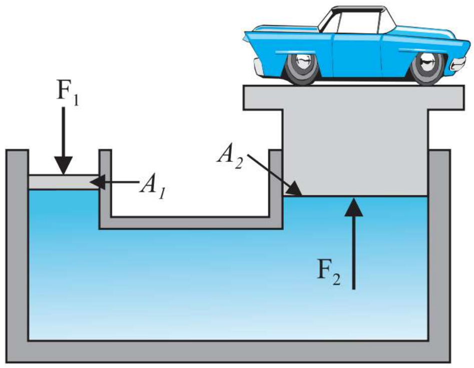


Fig 10.6 *(b) Schematic diagram illustrating the principle behind the hydraulic lift, a device used to lift heavy loads.*

> **Example 10.5** Two syringes of different cross-sections (without needles) filled with water are connected with a tightly fitted rubber tube filled with water. Diameters of the smaller piston and larger piston are 1.0 cm and 3.0 cm respectively. (a) Find the force exerted on the larger piston when a force of 10 N is applied to the smaller piston. (b) If the smaller piston is pushed in through 6.0 cm, how much does the larger piston move out?

Answer (a) Since pressure is transmitted undiminished throughout the fluid,

$$F_2 = \frac{A_2}{A_1} F_1 = \frac{\pi \left(3/2 \times 10^{-2}\,\text{m}\right)^2}{\pi \left(1/2 \times 10^{-2}\,\text{m}\right)^2} \times 10\,\text{N}$$

$$= 90\ \text{N}$$

(b) Water is considered to be perfectly incompressible. Volume covered by the movement of smaller piston inwards is equal to volume moved outwards due to the larger piston.

$$L_1 A_1 = L_2 A_2$$

$$L_2 = \frac{A_1}{A_2} L_1 = \frac{\pi \left(1/2 \times 10^{-2}\,\text{m}\right)^2}{\pi \left(3/2 \times 10^{-2}\,\text{m}\right)^2} \times 6 \times 10^{-2}\,\text{m}$$

$$\simeq 0.67 \times 10^{-2}\,\text{m} = 0.67\ \text{cm}$$

Note, atmospheric pressure is common to both pistons and has been ignored.

> **Example 10.6** In a car lift compressed air exerts a force F_1 on a small piston having a radius of 5.0 cm. This pressure is transmitted to a second piston of radius 15 cm (Fig 10.7). If the mass of the car to be lifted is 1350 kg, calculate F_1. What is the pressure necessary to accomplish this task? ($g = 9.8\ \text{ms}^{-2}$).

Answer Since pressure is transmitted undiminished throughout the fluid,

$$F_1 = \frac{A_1}{A_2} F_2 = \frac{\pi (5 \times 10^{-2}\,\text{m})^2}{\pi (15 \times 10^{-2}\,\text{m})^2} (1350\,\text{kg} \times 9.8\,m\,s^{-2})$$

$$= 1470\ \text{N}$$

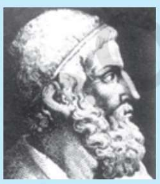
$$\approx 1.5 \times 10^3\ \text{N}$$

The air pressure that will produce this force is

$$P = \frac{F_1}{A_1} = \frac{1.5 \times 10^3\,\text{N}}{\pi \left(5 \times 10^{-2}\right)^2 \text{m}} = 1.9 \times 10^5\,\text{Pa}$$

This is almost double the atmospheric pressure.

Hydraulic brakes in automobiles also work on the same principle. When we apply a little force on the pedal with our foot the master piston



Archimedes (287–212 B.C.)

Archimedes was a Greek philosopher, mathematician, scientist and engineer. He invented the catapult and devised a system of pulleys and levers to handle heavy loads. The king of his native city Syracuse, Hiero II, asked him to determine if his gold crown was alloyed with some cheaper metal, such as silver without damaging the crown. The partial loss of weight he experienced while lying in his bathtub suggested a solution to him. According to legend, he ran naked through the streets of Syracuse, exclaiming "Eureka, eureka!", which means "I have found it, I have found it!"

moves inside the master cylinder, and the pressure caused is transmitted through the brake oil to act on a piston of larger area. A large force acts on the piston and is pushed down expanding the brake shoes against brake lining. In this way, a small force on the pedal produces a large retarding force on the wheel. An important advantage of the system is that the pressure set up by pressing pedal is transmitted equally to all cylinders attached to the four wheels so that the braking effort is equal on all wheels.

10.3 STREAMLINE FLOW

So far we have studied fluids at rest. The study of the fluids in motion is known as fluid dynamics. When a water tap is turned on slowly, the water flow is smooth initially, but loses its smoothness when the speed of the outflow is increased. In studying the motion of fluids, we focus our attention on what is happening to various fluid particles at a particular point in space at a particular time. The flow of the fluid is said to be **steady** if at any given point, the velocity of each passing fluid particle remains constant in time. This does not mean that the velocity at different points in space is same. The velocity of a particular particle may change as it moves from one point to another. That is, at some other point the particle may have a different velocity, but every other particle which passes the second point behaves exactly as the previous particle that has just passed that point. Each particle follows a smooth path, and the paths of the particles do not cross each other.

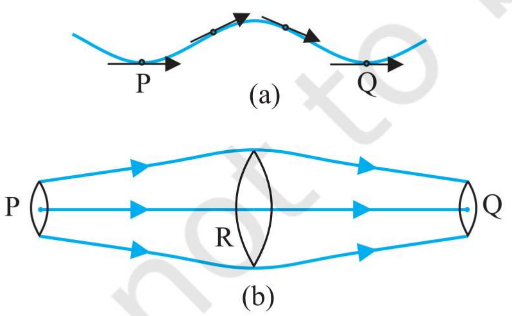


***Fig. 10.7** The meaning of streamlines. (a) A typical trajectory of a fluid particle. (b) A region of streamline flow.*

The path taken by a fluid particle under a steady flow is a **streamline**. It is defined as a curve whose tangent at any point is in the direction of the fluid velocity at that point. Consider the path of a particle as shown in Fig.10.7 (a), the curve describes how a fluid particle moves with time. The curve PQ is like a permanent map of fluid flow, indicating how the fluid streams. No two streamlines can cross, for if they do, an oncoming fluid particle can go either one way or the other and the flow would not be steady. Hence, in steady flow, the map of flow is stationary in time. How do we draw closely spaced streamlines ? If we intend to show streamline of every flowing particle, we would end up with a continuum of lines. Consider planes perpendicular to the direction of fluid flow e.g., at three points P, R and Q in Fig.10.7 (b). The plane pieces are so chosen that their boundaries be determined by the same set of streamlines. This means that number of fluid particles crossing the surfaces as indicated at P, R and Q is the same. If area of cross-sections at these points are A_P,A_R and A_Q and speeds of fluid particles are v_P, v_R and v_Q, then mass of fluid Δm_P crossing at A_P in a small interval of time Δt is $\rho_P A_P v_P \Delta t$. Similarly mass of fluid Δm_R flowing or crossing at A_R in a small interval of time Δt is $\rho_R A_R v_R \Delta t$ and mass of fluid Δm_Q is $\rho_Q A_Q v_Q \Delta t$ crossing at A_Q. The mass of liquid flowing out equals the mass flowing in, holds in all cases. Therefore,

$$\rho_P A_P v_P \Delta t = \rho_R A_R v_R \Delta t = \rho_Q A_Q v_Q \Delta t \qquad (10.9)$$

For flow of incompressible fluids

$$\rho_P = \rho_R = \rho_Q$$

Equation (10.9) reduces to

$$A_P v_P = A_R v_R = A_Q v_Q \qquad (10.10)$$

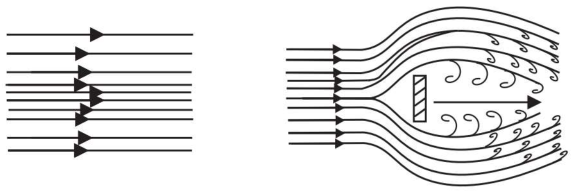
which is called the **equation of continuity** and it is a statement of conservation of mass in flow of incompressible fluids. In general

$$Av = \text{constant} \qquad (10.11)$$

Av gives the volume flux or flow rate and remains constant throughout the pipe of flow. Thus, at narrower portions where the streamlines are closely spaced, velocity increases and its vice versa. From (Fig 10.7b) it is clear that $A_R > A_Q$ or $v_R < v_Q$, the fluid is accelerated while passing from R to Q. This is associated with a change in pressure in fluid flow in horizontal pipes.

Steady flow is achieved at low flow speeds. Beyond a limiting value, called critical speed, this flow loses steadiness and becomes **turbulent**. One sees this when a fast flowing stream encounters rocks, small foamy whirlpool-like regions called 'white water rapids are formed.

Figure 10.8 displays streamlines for some typical flows. For example, Fig. 10.8(a) describes a laminar flow where the velocities at different points in the fluid may have different magnitudes but their directions are parallel. Figure 10.8 (b) gives a sketch of turbulent flow.



(a) (b)

Fig. 10.8 *(a) Some streamlines for fluid flow. (b) A jet of air striking a flat plate placed perpendicular to it. This is an example of turbulent flow.*

10.4 BERNOULLI'S PRINCIPLE

Fluid flow is a complex phenomenon. But we can obtain some useful properties for steady or streamline flows using the conservation of energy.

Consider a fluid moving in a pipe of varying cross-sectional area. Let the pipe be at varying heights as shown in Fig. 10.9. We now suppose that an incompressible fluid is flowing through the pipe in a steady flow. Its velocity must change as a consequence of equation of continuity. A force is required to produce this acceleration, which is caused by the fluid surrounding it, the pressure must be different in different regions. Bernoulli's equation is a general expression that relates the pressure difference between two points in a pipe to both velocity changes (kinetic energy change) and elevation (height) changes (potential energy change). The Swiss Physicist Daniel Bernoulli developed this relationship in 1738.

Consider the flow at two regions 1 (i.e., BC) and 2 (i.e., DE). Consider the fluid initially lying between B and D. In an infinitesimal time interval Δt, this fluid would have moved. Suppose v_1 is the speed at B and v_2 at D, then fluid initially at B has moved a distance $v_1\Delta t$ to C ($v_1\Delta t$ is small enough to assume constant cross-section along BC). In the same interval Δt the fluid initially at D moves to E, a distance equal to $v_2\Delta t$. Pressures P_1 and P_2 act as shown on the plane faces of areas A_1 and A_2 binding the two regions. The work done on the fluid at left end (BC) is $W_1 = P_1A_1(v_1\Delta t) = P_1\Delta V$. Since the same volume ΔV passes through both the regions (from the equation of continuity) the work done by the fluid at the other end (DE) is $W_2 = P_2A_2(v_2\Delta t) = P_2\Delta V$ or, the work done on the fluid is $-P_2\Delta V$. So the total work done on the fluid is

$$W_1 - W_2 = (P_1 - P_2)\,\Delta V$$

Part of this work goes into changing the kinetic energy of the fluid, and part goes into changing the gravitational potential energy. If the density of the fluid is ρ and $\Delta m = \rho A_1 v_1 \Delta t = \rho\Delta V$ is the mass passing through the pipe in time Δt, then change in gravitational potential energy is

$$\Delta U = \rho g \Delta V\,(h_2 - h_1)$$

The change in its kinetic energy is

$$\Delta K = \rho\,\Delta V\,(v_2^2 - v_1^2)$$

We can employ the work – energy theorem (Chapter 6) to this volume of the fluid and this yields

$$(P_1 - P_2)\,\Delta V = \left(\frac{1}{2}\right)\rho\,\Delta V\,(v_2^2 - v_1^2) + \rho g \Delta V\,(h_2 - h_1)$$

We now divide each term by ΔV to obtain

$$(P_1 - P_2) = \left(\frac{1}{2}\right)\rho\,(v_2^2 - v_1^2) + \rho g\,(h_2 - h_1)$$

Daniel Bernoulli (1700–1782)

Daniel Bernoulli was a Swiss scientist and mathematician, who along with Leonard Euler had the distinction of winning the French Academy prize for mathematics 10 times. He also studied medicine and served as a professor of anatomy and botany for a while at Basle, Switzerland. His most well-known work was in hydrodynamics, a subject he developed from a single principle: the conservation of energy. His work included calculus, probability, the theory of vibrating strings, and applied mathematics. He has been called the founder of mathematical physics.