

If p is nearly equal to q and $n > 1$, show that $\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \left(\frac{p}{q}\right)^{1/n}$. Hence find the approximate value of $\left(\frac{99}{101}\right)^{1/6}$

SOLUTION :

Given that p is nearly equal to q , hence assume $p = q + h$, where h is very small.

Then,

$$\begin{aligned} \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} &= \frac{(n+1)(q+h)+(n-1)q}{(n-1)(q+h)+(n+1)q} \\ &= \frac{2nq+(n+1)h}{2nq+(n-1)h} = \frac{1+\left(\frac{n+1}{2nq}\right)h}{1+\left(\frac{n-1}{2nq}\right)h}, \quad n > 1 \\ &= \left[1 + \left(\frac{n+1}{2nq}\right)h\right] \left[1 + \left(\frac{n-1}{2nq}\right)h\right]^{-1} \\ &= \left[1 + \left(\frac{n+1}{2nq}\right)h\right] \left[1 - \left(\frac{n-1}{2nq}\right)h + \frac{(-1)(-1-1)}{2!} \left(\frac{n-1}{2nq}\right)^2 h^2 + \dots\right] \end{aligned}$$

Since ' h ' is very small, hence h^2, h^3, \dots are neglected.

$$\text{Therefore, } \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \left[1 + \left(\frac{n+1}{2nq}\right)h\right] \left[1 - \left(\frac{n-1}{2nq}\right)h\right] = 1 - \frac{(n-1)}{2nq}h + \frac{(n+1)}{2nq}h - \frac{(n+1)(n-1)}{4n^2q^2}h^2$$

$$= 1 + \frac{h}{nq} \quad (\text{Again neglecting } h^2 \text{ term})$$

$$= \left[1 + \frac{h}{q}\right]^{\frac{1}{n}} = \left[\frac{q+h}{q}\right]^{\frac{1}{n}} = \left[\frac{p}{q}\right]^{\frac{1}{n}}$$

$$\therefore \frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \left(\frac{p}{q}\right)^{\frac{1}{n}}$$

Put $p = 99$, $q = 101$ and $n = 6$

$$\left(\frac{99}{101}\right)^{\frac{1}{6}} = \frac{7 \times 99 + 5 \times 101}{5 \times 99 + 7 \times 101} = \frac{693 + 505}{495 + 707} = \frac{1198}{1202}$$