Question: -

$\lim_{x \to 1-} \dfrac{\sqrt{\pi} - \sqrt{2\sin^{-1} x}}{\sqrt{1-x}}$ is equal to

(2019 Main, 12 Jan II)

(a) $\sqrt{\dfrac{\pi}{2}}$ (b) $\sqrt{\dfrac{2}{\pi}}$ (c) $\sqrt{\pi}$ (d) $\dfrac{1}{\sqrt{2\pi}}$

Solution: -

Let $L = \lim_{x \to 1^-} \dfrac{\sqrt{\pi} - \sqrt{2\sin^{-1} x}}{\sqrt{1-x}}$, then

$$L = \lim_{x \to 1^-} \frac{\sqrt{\pi} - \sqrt{2\sin^{-1} x}}{\sqrt{1-x}} \times \frac{\sqrt{\pi} + \sqrt{2\sin^{-1} x}}{\sqrt{\pi} + \sqrt{2\sin^{-1} x}}$$

[on rationalization]

$$= \lim_{x \to 1^-} \frac{\pi - 2\sin^{-1} x}{\sqrt{1-x}} \times \frac{1}{\sqrt{\pi} + \sqrt{2\sin^{-1} x}}$$

$$= \lim_{x \to 1^-} \frac{\pi - 2\left(\frac{\pi}{2} - \cos^{-1} x\right)}{\sqrt{1-x}} \times \frac{1}{\sqrt{\pi} + \sqrt{2\sin^{-1} x}}$$

$\left[\because \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}\right]$

$$= \lim_{x \to 1^-} \frac{2\cos^{-1} x}{\sqrt{1-x}} \times \lim_{x \to 1^-} \frac{1}{\sqrt{\pi} + \sqrt{2\sin^{-1} x}}$$

$$= \frac{1}{2\sqrt{\pi}} \lim_{x \to 1^-} \frac{2\cos^{-1} x}{\sqrt{1-x}} \qquad \left[\because \lim_{x \to 1^-} \sin^{-1} x = \frac{\pi}{2}\right]$$

Put $x = \cos\theta$, then as $x \to 1^-$, therefore $\theta \to 0^+$

Now, $L = \dfrac{1}{2\sqrt{\pi}} \lim_{\theta \to 0^+} \dfrac{2\theta}{\sqrt{1-\cos\theta}}$

$$= \frac{1}{2\sqrt{\pi}} \lim_{\theta \to 0^+} \frac{2\theta}{\sqrt{2}\sin\left(\frac{\theta}{2}\right)} \qquad \left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{2\sqrt{\pi}} \cdot \sqrt{2} \lim_{\theta \to 0^+} \frac{2 \cdot \left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{1}{2\sqrt{\pi}} 2\sqrt{2} = \sqrt{\frac{2}{\pi}} \qquad \left[\because \lim_{x \to 0^+} \frac{\theta}{\sin\theta} = 1\right]$$