Question: -

Let A, B, C be three angles such that $A = \frac{\pi}{4}$ and $\tan B, \tan C = p$. Find all positive values of p such that A, B, C are the angles of triangle. **(1997C, 5M)**

Solution: -

Since, $A + B + C = \pi$

$\Rightarrow \quad B + C = \pi - \pi/4 = 3\pi/4$ ...(i)

$[\because A = \pi/4, \text{given}]$

$\therefore \quad 0 < B, C < 3\pi/4$

Also, given $\tan B \cdot \tan C = p$

$$\Rightarrow \quad \frac{\sin B \cdot \sin C}{\cos B \cdot \cos C} = \frac{p}{1}$$

$$\Rightarrow \quad \frac{\sin B \cdot \sin C + \cos B \cos C}{\sin B \cdot \sin C - \cos B \cdot \cos C} = \frac{p+1}{p-1}$$

$$\Rightarrow \quad \frac{\cos (B - C)}{\cos (B + C)} = \frac{1+p}{1-p}$$

$$\Rightarrow \quad \cos (B - C) = -\frac{(1+p)}{\sqrt{2}\,(1-p)} \quad \text{...(ii)}$$

$[\because B + C = 3\pi/4]$

Since, B or C can vary from 0 to $3\pi/4$

$\therefore \quad 0 \le B - C < 3\pi/4$

$$\Rightarrow \quad -\frac{1}{\sqrt{2}} < \cos (B - C) \le 1 \quad \text{...(iii)}$$

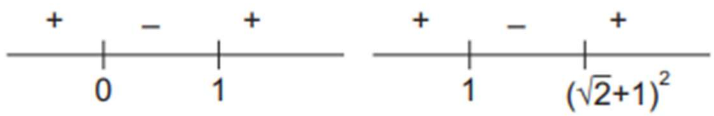
From Eqs. (ii) and (iii), $-\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}\,(p-1)} \le 1$

$$\Rightarrow \quad -\frac{1}{\sqrt{2}} < \frac{1+p}{\sqrt{2}\,(p-1)} \quad \text{and} \quad \frac{1+p}{\sqrt{2}\,(p-1)} \le 1$$

$$\Rightarrow \quad \frac{1+p}{p-1} + 1 \ge 0 \quad \text{and} \quad \frac{1 + p - \sqrt{2}\,p + \sqrt{2}}{\sqrt{2}(p-1)} \le 0$$

$$\Rightarrow \qquad \frac{2p}{p-1} \geq 0 \quad \text{and} \quad \frac{(1-\sqrt{2})\left(p - \dfrac{1+\sqrt{2}}{1-\sqrt{2}}\right)}{\sqrt{2}(p-1)} \leq 0$$

$$\Rightarrow \qquad \frac{2p}{p-1} > 0 \quad \text{and} \quad \frac{(p-(\sqrt{2}+1)^2)}{(p-1)} \geq 0$$



$\Rightarrow \qquad (p < 0 \text{ or } p > 1)$

and $\qquad (p < 1 \text{ or } p > (\sqrt{2}+1)^2)$

On combining above expressions, we get

$$p < 0 \text{ or } p \geq (\sqrt{2}+1)^2$$

i.e. $\qquad p \in (-\infty, 0) \cup [(\sqrt{2}+1)^2, \infty)$

or $\qquad p \in (-\infty, 0) \cup [3 + 2\sqrt{2}, \infty)$