

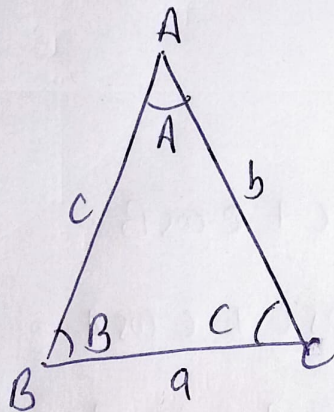
①

→ Relation between the sides and Angles of triangle:-

In a  $\triangle ABC$ , angles are denoted by  $A$ ,  $B$ , and  $C$   
 the lengths of corresponding sides opposite to these  
 angles are denoted by  $a$ ,  $b$  and  $c$  respectively.

Area and perimeter of triangle denoted by  
 $\Delta$  and  $2s$  respectively.

Semi-perimeter of the triangle is,  $s = \frac{a+b+c}{2}$



### Sine Rule-

In any  $\triangle ABC$ , the sines of the angles are proportional  
 to the lengths of the opposite side, i.e.,

$$\frac{\sin A}{a} = \frac{\sin b}{b} = \frac{\sin C}{c} = k$$

Also,  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$  (say).

Then,  $a = k \sin A$ ,  $b = k \sin B$ ,  $c = k \sin C$

## Cosine Rule -

(2)

In Any  $\triangle ABC$ , cosine of an angle can express in terms of sides.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

## Projection Rule -

In  $\triangle ABC$ ,

$$a = b \cos C + c \cos B$$

$$b = a \cos C + c \cos A$$

$$c = a \cos B + b \cos A$$

## Napier's Rule -

In Any  $\triangle ABC$ ,

$$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$$

⇒ Trigonometric Ratios of Half Angles of any triangle:

In a  $\triangle ABC$ , if the sides of the triangle are  $a, b, c$  and corresponding angles are  $A, B, C$  respectively and  $s$  is the semi-perimeter,

then, (1) 
$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

(2) 
$$\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

(3) 
$$\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

(4) 
$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

(5) 
$$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$$

(6) 
$$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

(7) 
$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

(8) 
$$\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

(9) 
$$\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Q3 Area of triangles-

In  $\Delta ABC$ , if the sides of the triangle are  $a, b, c$  and corresponding angles are  $A, B, C$  respectively, then area of triangle;

When two sides and Angles between them is known.

$$\Delta = \frac{1}{2} ab \sin C$$

$$\Delta = \frac{1}{2} bc \sin A$$

$$\Delta = \frac{1}{2} ca \sin B$$

When one side and corresponding Angles are known

$$\Delta = \frac{c^2 \sin A \sin B}{2 \sin C}$$

$$\Delta = \frac{a^2 \sin B \sin C}{2 \sin A}$$

$$\Delta = \frac{b^2 \sin C \sin A}{2 \sin B}$$

When all the three sides are known

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

It is known as Heron's formula.

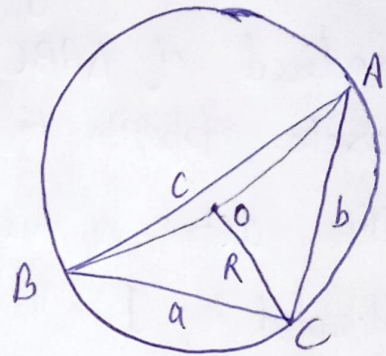
## → Different Types of circle Connected with triangles - (5)

### (1) Circumcircle of a triangle -

The circle passing through the vertices of a  $\triangle ABC$  is called circumcircle. Its radius  $R$  is called the circumradius and its centre is known as circumcentre. Circumcentre is the point of intersection of perpendicular bisectors of the sides.

$$1. R = \frac{a}{2\sin A} = \frac{c}{2\sin C}$$

$$2. R = \frac{abc}{4\Delta}$$



### (2) Incircle of a triangle -

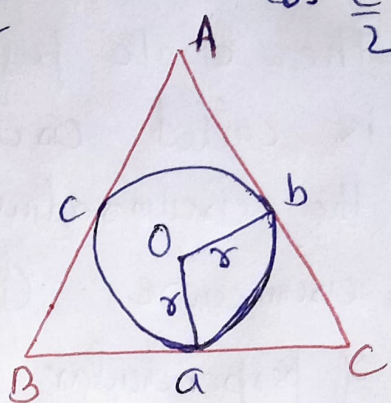
The circle touching all the sides of a triangle internally is known as an incircle of the triangle. Its centre is called incentre and its radius  $r$  is called the inradius of the circle. Incenter is the point of intersection of bisectors of the angles of the triangle.

$$1. r = \frac{\Delta}{s}$$

$$2. r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$$

$$\textcircled{3} \quad r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} = \frac{b \sin \frac{A}{2} \sin \frac{C}{2}}{\cos \frac{B}{2}} = \frac{c \sin \frac{B}{2} \sin \frac{A}{2}}{\cos \frac{C}{2}}$$

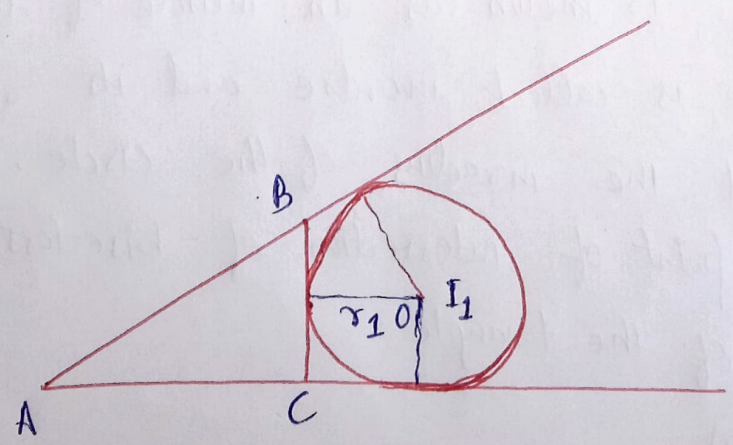
$$\textcircled{4} \quad r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$



Exscribed circles of a triangle-

The circles touching BC and the two sides AB and AC produced of  $\triangle ABC$  externally is called the exscribed circle opposite to A. Its radius is denoted by  $r_1$ .

The centre of this circle is known as excentre and denoted by  $I_1$ . It is the point where the external bisectors of the angle B and C and the internal bisector of the the angle A meet.



Similarly,  $r_2$  and  $r_3$  denote the radii of the escribed circles opposite to the angle B and C respectively and excentres are denoted by  $I_2$  and  $I_3$ .  $r_1, r_2, r_3$  are ~~the~~ called the exradii of  $\triangle ABC$ . These,

$$1. r_1 = \frac{\Delta}{s-a} = s \tan \frac{A}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$$

$$2. r_2 = \frac{\Delta}{s-b} = s \tan \frac{B}{2} = 4R \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2} = \frac{b \cos \frac{A}{2} \cos \frac{C}{2}}{\cos \frac{B}{2}}$$

$$3. r_3 = \frac{\Delta}{s-c} = s \tan \frac{C}{2} = 4R \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} = \frac{c \cos \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{C}{2}}$$

⇒ Orthocentre of a triangle-

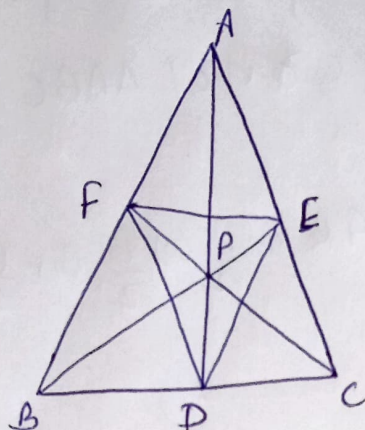
In a  $\triangle ABC$ , AD, BE, and CF are perpendiculars from the vertices A, B and C respectively to the opposite sides. These three perpendiculars are concurrent at the point P which is called the orthocentre of the  $\triangle ABC$ .

The  $\triangle DEF$  is called the pedal triangle of  $\triangle ABC$ .

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①  $PA = 2R \cos A$ ,  $PB = 2R \cos B$ ,  $PC = 2R \cos C$

②  $PD = 2R \cos B \cos C$ ,  $PE = 2R \cos C \cos A$ ,  $PF = 2R \cos A \cos B$



⇒ Regular polygon:-

A regular polygon is a polygon which has all its sides as well its angles are equal. The circle passing through all the vertices of a regular polygon is called its circumscribed circle and the circle which touches all the sides of a regular polygon is called its inscribed circle.

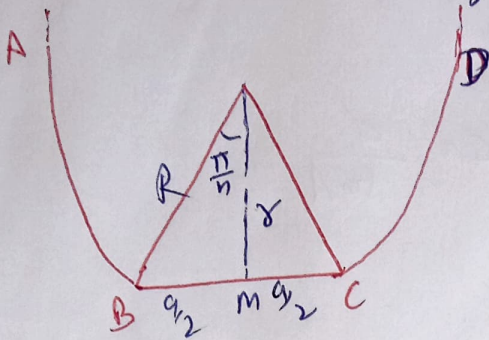
Radius of circumcircle,  $R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$

and radius of incircle,  $r = \frac{a}{2} \cot \frac{\pi}{n}$

where,  $a$  is the length of the side of a regular polygon.



Area of a regular polygon =  $\frac{1}{4} na^2 \cot \frac{\pi}{n}$



=  $nr^2 \tan \frac{\pi}{n}$

=  $\frac{n}{2} R^2 \sin \frac{2\pi}{n}$

0) Solutions of triangles

The three sides  $a, b$  and  $c$  & three angle  $A, B, C$  are called the elements of the  $ABC$ . When any three of these six elements (except all the three angles) of a triangle is given, the triangle is known as completely. That is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

Solution of right angled triangle,

Let the triangle be right angled at  $C$ , then

(i) When two sides are given - If  $a, b \rightarrow$  given

then  $\tan A = \frac{a}{b}$ ,  $B = 90 - A$ ,  $c = \frac{a}{\sin A}$

if  $a, c \rightarrow$  given, then  $\sin A = \frac{a}{c}$ ,  $b = c \cos A$ ,  $B = 90 - A$

(ii) When one side and an acute angle are given,

If  $a, A$  are given.

$$\text{then } B = 90^\circ - A, \quad b = a \cot A, \quad c = \frac{a}{\sin A}$$

if  $c, A \rightarrow$  given

$$\text{then, } B = 90^\circ - A, \quad a = c \sin A, \quad b = c \cos A$$

### solution of a triangle in General

(i) If the three sides  $a, b, c$  are given,

then, angle  $A$  is obtained from,

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

(ii) If two sides  $b$  and  $c$  and the included angle  $A$  are given, then

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} \quad \text{gives} \quad \frac{B-C}{2}$$

Also,  $\frac{B+C}{2} = 90^\circ - \frac{A}{2}$ , so that  $B$ , and  $C$  can be evaluated.

The third side is obtained either by sine rule or cosine rule.

(iii) When one side 'a' and two angles ( $\angle A$  &  $\angle B$ ) are given, then  $\angle C = 180^\circ - (\angle A + \angle B)$  and the other sides can be determined by sine rule. (11)

(iv) When two sides and angle opposite to one of them be given.

In this case, the triangle is not always uniquely determined. It is quite possible to have no triangle, one triangle and two triangles with this type of data. So, it is called an ambiguous ~~state~~ case.

By cosine law,  $b^2 = c^2 + a^2 - 2ac \cos B$

From this,

$$\Rightarrow a = c \cos B \pm \sqrt{b^2 - c^2 \sin^2 B}$$

This equation helps us to determine a, when b, c and B are being given. Now the following cases arise.

Case 1 - When  $b < c \sin B$ , then a is imaginary and so there is no solution.

Case 2 - When  $b = c \sin B$   
then  $a = c \cos B$ , so there is unique solution.

Also,  $b = c \sin B$

$$\frac{k \sin B}{\sin B} = \frac{k \sin C}{\sin C} \Rightarrow \sin C = 1$$

$$\angle C = 90^\circ$$

, right angled triangle in this case.

Case III,

$$b > c \sin B$$

then,  $b^2 - c^2 \sin^2 B > 0$

Then, there are two solutions given by,

$$a = c \cos B + \sqrt{b^2 - c^2 \sin^2 B}$$

$$a = c \cos B - \sqrt{b^2 - c^2 \sin^2 B}$$

Now if  $B$  is an ~~acute~~ acute angle, then there are two triangles provided that  $c > b > c \sin B$

and if  $B$  is an obtuse angle, then there is only one triangle provided that  $b > c$ .