Question: -

In a ΔXYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z respectively and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the ΔXYZ is $\frac{8\pi}{3}$, then **(2016 Adv.)**

(a) area of the ΔXYZ is $6\sqrt{6}$

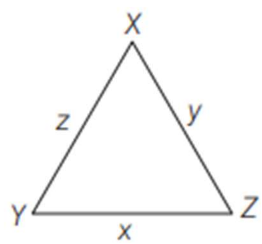
(b) the radius of circumcircle of the ΔXYZ is $\frac{35}{6}\sqrt{6}$

(c) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$ (d) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

Solution: -

Given a ΔXYZ, where $2s = x + y + z$

and $$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$$



$\therefore$ $$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$$
$$= \frac{3s-(x+y+z)}{4+3+2} = \frac{s}{9}$$

or $$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{s}{9} = \lambda \text{ (let)}$$

$\Rightarrow$ $s = 9\lambda,\ s = 4\lambda + x, s = 3\lambda + y$

and $s = 2\lambda + z$

$\therefore$ $s = 9\lambda,\ x = 5\lambda,\ y = 6\lambda,\ z = 7\lambda$

Now, $\Delta = \sqrt{s(s-x)(s-y)(s-z)}$

[Heron's formula]

$$= \sqrt{9\lambda \cdot 4\lambda \cdot 3\lambda \cdot 2\lambda} = 6\sqrt{6}\lambda^2 \quad \text{...(i)}$$

Also, $$\pi r^2 = \frac{8\pi}{3}$$

$\Rightarrow$ $$r^2 = \frac{8}{3} \quad \text{...(ii)}$$

and $$R = \frac{xyz}{4\Delta} = \frac{(5\lambda)(6\lambda)(7\lambda)}{4 \cdot 6\sqrt{6}\lambda^2} = \frac{35\lambda}{4\sqrt{6}} \quad \text{...(iii)}$$

Now, $$r^2 = \frac{8}{3} = \frac{\Delta^2}{S^2} = \frac{216\lambda^4}{81\lambda^2}$$

$\Rightarrow$ $$\frac{8}{3} = \frac{8}{3}\lambda^2 \quad \text{[from Eq. (ii)]}$$

$\Rightarrow$ $$\lambda = 1$$

(a) $\Delta XYZ = 6\sqrt{6}\lambda^2 = 6\sqrt{6}$

$\therefore$ Option (a) is correct.

(b) Radius of circumcircle $(R) = \frac{35}{4\sqrt{6}}\lambda = \frac{35}{4\sqrt{6}}$

$\therefore$ Option (b) is incorrect.

(c) Since, $$r = 4R\sin\frac{X}{2}\cdot\sin\frac{Y}{2}\cdot\sin\frac{Z}{2}$$

$\Rightarrow$ $$\frac{2\sqrt{2}}{\sqrt{3}} = 4\cdot\frac{35}{4\sqrt{6}}\sin\frac{X}{2}\cdot\sin\frac{Y}{2}\cdot\sin\frac{Z}{2}$$

$\Rightarrow$ $$\frac{4}{35} = \sin\frac{X}{2}\cdot\sin\frac{Y}{2}\cdot\sin\frac{Z}{2}$$

$\therefore$ Option (c) is correct.

(d) $\sin^2\left(\frac{X+Y}{2}\right) = \cos^2\left(\frac{Z}{2}\right)$

as $\frac{X+Y}{2} = 90° - \frac{Z}{2} = \frac{s(s-z)}{xy} = \frac{9\times 2}{5\times 6} = \frac{3}{5}$

$\therefore$ Option (d) is correct.