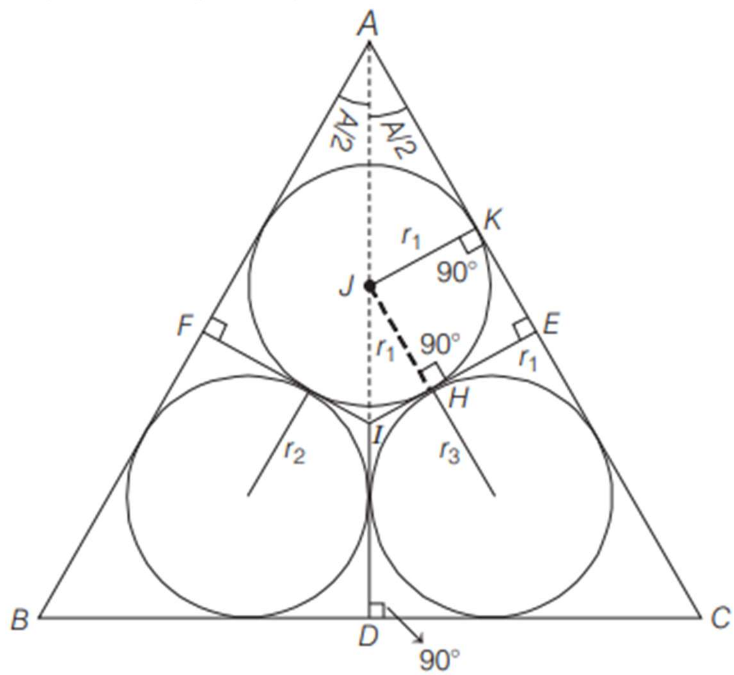
Question: -

Let ABC be a triangle with incentre I and inradius r. Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA and AB, respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEID$ respectively, then prove that

$$\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}.$$ **(2000 3M)**

Solution: -

The quadrilateral $HEKJ$ is a square, because all four angles are right angles and $JK = JH$.



Therefore, $HE = JK = r_1$ and $IE = r$ [given]

$\Rightarrow$ $IH = r - r_1$

Now, in right angled ΔIHJ,

$$\angle JIH = \pi/2 - A/2$$

$[\because \angle IEA = 90°, \angle IAE = A/2 \text{ and } \angle JIH = \angle AIE]$

In ΔJIH,

$$\tan\left(\frac{\pi}{2} - \frac{A}{2}\right) = \frac{r_1}{r - r_1} \Rightarrow \cot\frac{A}{2} = \frac{r_1}{r - r_1}$$

Similarly, $\cot\frac{B}{2} = \frac{r_2}{r - r_2}$ and $\cot\frac{C}{2} = \frac{r_3}{r - r_3}$

On adding above results, we get

$$\cot A/2 + \cot B/2 + \cot C/2$$
$$= \cot A/2 \cot B/2 \cot C/2$$

$$\Rightarrow \frac{r_1}{r - r_1} + \frac{r_2}{r - r_2} + \frac{r_3}{r - r_3} = \frac{r_1 r_2 r_3}{(r - r_{1)}\, (r - r_2)\, (r - r_3)}$$