

5) Prove that
$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Solution:
$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} a^2(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & c^2 \\ a^2 & b^2 & c(c^2+1) \end{vmatrix}$$

[Multiplying C_1, C_2, C_3 by a, b, c respectively]

$$= \frac{abc}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

[Taking common a, b, c from R_1, R_2, R_3 respectively]

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \quad [C_1 \rightarrow C_1 + C_2 + C_3]$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

[Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$]

$$= (1+a^2+b^2+c^2)$$