5) Solve the following linear equation by rank method.

$$2x+5y+7z=52\ ,\quad x+y+z=9,\quad 2x+y-z=0$$

Solution : The equation can be rewritten as $AX=B$

where $A=\begin{bmatrix}2&5&7\\1&1&1\\2&1&-1\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}52\\9\\0\end{bmatrix}$

$$[A,B]\sim\left[\begin{array}{ccc|c}2&5&7&52\\1&1&1&9\\2&1&-1&0\end{array}\right]$$

$$R_1\leftrightarrow R_2\quad\left[\begin{array}{ccc|c}1&1&1&9\\2&5&7&52\\2&1&-1&0\end{array}\right]$$

$R_2\to R_2-2R_1$

$R_3\to R_3-2R_1$

$$\left[\begin{array}{ccc|c}1&1&1&9\\0&3&5&34\\0&-1&-3&-18\end{array}\right]$$

$R_3\to 3R_3$

$R_3\to R_3+R_1$

$$\left[\begin{array}{ccc|c}1&1&1&9\\0&3&5&34\\0&0&-4&-20\end{array}\right]$$

Number of non-zero rows $=3$.

$\text{Rank}(A)=\text{Rank}([A|B])=3$. The system is consistent and it has unique solution.

$\therefore\ x+y+z=9$ —— ①

$3y+5z=34$ —— ②

$-4z=-20\ \Rightarrow\ z=5.$

Using $z=5$ in ②, $3y+25=34\Rightarrow 3y=9\Rightarrow y=3.$

Using $z=5$ and $y=3$ in ①, $x+8=9$

$\Rightarrow x=1.$

So the solution is $(1,3,5)$.