4.7 Applications of Determinants and Matrices

In this section, we shall discuss application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

Consistent system A system of equations is said to be *consistent* if its solution (one or more) exists.

Inconsistent system A system of equations is said to be *inconsistent* if its solution does not exist.

Note In this chapter, we restrict ourselves to the system of linear equations having unique solutions only.

4.7.1 *Solution of system of linear equations using inverse of a matrix*

Let us express the system of linear equations as matrix equations and solve them using inverse of the coefficient matrix.

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then, the system of equations can be written as, AX = B, i.e.,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I If A is a nonsingular matrix, then its inverse exists. Now

$$AX = B$$

or $$A^{-1}(AX) = A^{-1}B \quad \text{(premultiplying by } A^{-1}\text{)}$$

or $$(A^{-1}A)X = A^{-1}B \quad \text{(by associative property)}$$

or $$IX = A^{-1}B$$

or $$X = A^{-1}B$$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

Case II If A is a singular matrix, then $|A| = 0$.

In this case, we calculate (*adj* A) B.

If (*adj* A) B $\neq$ O, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If (*adj* A) B = O, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

Example 27 Solve the system of equations

$$2x + 5y = 1$$
$$3x + 2y = 7$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now, $|A| = -11 \neq 0$, Hence, A is nonsingular matrix and so has a unique solution.

Note that $$A^{-1} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Therefore $$X = A^{-1}B = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence $x = 3, y = -1$

Example 28 Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

We see that

$$|A| = 3\,(2 - 3) + 2(4 + 4) + 3\,(-6 - 4) = -17 \neq 0$$