Question: - Prove that the line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$.

Sol. The equation of any normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$ax \cos\phi + by \cot\phi = a^2 + b^2$$

or $$ax \cos\phi + by \cot\phi - (a^2 + b^2) = 0 \qquad \text{...(i)}$$

The straight line $lx + my - n = 0$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then Eq. (i) and $lx + my - n = 0$ represent the same line

$$\frac{a\cos\phi}{l} = \frac{b\cot\phi}{m} = \frac{(a^2 + b^2)}{n}$$

or $$\sec\phi = \frac{na}{l(a^2 + b^2)} \text{ and } \tan\phi = \frac{nb}{m(a^2 + b^2)}$$

$\because$ $$\sec^2\phi - \tan^2\phi = 1$$

$\therefore$ $$\frac{n^2a^2}{l^2(a^2 + b^2)^2} - \frac{n^2b^2}{m^2(a^2 + b^2)^2} = 1$$

$\Rightarrow$ $$\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$