Question: – Find the equation and the length of the common tangents to hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad and \quad \frac{y^2}{a^2}-\frac{x^2}{b^2}=1.$$

Sol. Tangent at $(a \sec \phi, b \tan \phi)$ on the 1st hyperbola is

$$\frac{x}{a}\sec \phi - \frac{y}{b}\tan \phi = 1 \qquad ...(i)$$

Similarly tangent at any point $(b \tan\theta, a \sec \theta)$ on 2nd hyperbola is

$$\frac{y}{a}\sec \theta - \frac{x}{b}\tan\theta = 1 \qquad ...(ii)$$

If Eqs. (i) and (ii) are common tangents then they should be identical. Comparing the coefficients of x and y

$\Rightarrow$ $$\frac{\sec\theta}{a} = -\frac{\tan\phi}{b}$$

or $$\sec\theta = -\frac{a}{b}\tan\phi \qquad ...(iii)$$

and $$-\frac{\tan\theta}{b} = \frac{\sec\phi}{a}$$

or $$\tan\theta = -\frac{b}{a}\sec\phi \qquad ...(iv)$$

$\because$ $$\sec^2\theta - \tan^2\theta = 1$$

$\Rightarrow$ $$\frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}\sec^2\phi = 1 \qquad \text{[from Eqs. (iii) and (iv)]}$$

or $$\frac{a^2}{b^2}\tan^2\phi - \frac{b^2}{a^2}(1+\tan^2\phi) = 1$$

or $$\left(\frac{a^2}{b^2}-\frac{b^2}{a^2}\right)\tan^2\phi = 1 + \frac{b^2}{a^2}$$

$\therefore$ $$\tan^2\phi = \frac{b^2}{a^2-b^2}$$

and $$\sec^2\phi = 1+\tan^2\phi = \frac{a^2}{a^2-b^2}$$

Hence, the points of contact are

$$\left\{\pm\frac{a^2}{\sqrt{(a^2-b^2)}}, \pm\frac{b^2}{\sqrt{(a^2-b^2)}}\right\}$$

and $$\left\{\mp\frac{b^2}{\sqrt{(a^2-b^2)}}, \mp\frac{a^2}{\sqrt{(a^2-b^2)}}\right\} \qquad \text{[from Eqs. (iii) and (iv)]}$$

Length of common tangent i.e. the distance between the above points is $\sqrt{2}\,\dfrac{(a^2+b^2)}{\sqrt{(a^2-b^2)}}$ and equation of common tangent on putting the values of $\sec\phi$ and $\tan\phi$ in Eqs. (i) is

$$\pm\frac{x}{\sqrt{(a^2-b^2)}} \mp \frac{y}{\sqrt{(a^2-b^2)}} = 1$$

or $$x \mp y = \pm\sqrt{(a^2-b^2)}$$