Chapter 2
Statics

The subject of statics often appears in later chapters in other books, after force and torque have been discussed. However, the way that force and torque are used in statics problems is fairly minimal, at least compared with what we'll be doing later in this book. Therefore, since we won't be needing much of the machinery that we'll be developing later on, I'll introduce here the bare minimum of force and torque concepts necessary for statics problems. This will open up a whole class of problems for us. But even though the underlying principles of statics are quick to state, statics problems can be unexpectedly tricky. So be sure to tackle a lot of them to make sure you understand things.

2.1 Balancing forces

A "static" setup is one where all the objects are motionless. If an object remains motionless, then Newton's second law, $F = ma$ (which we'll discuss in great detail in the next chapter), tells us that the total external force acting on the object must be zero. The converse is not true, of course. The total external force on an object is also zero if it moves with constant nonzero velocity. But we'll deal only with statics problems here. The whole goal in a statics problem is to find out what the various forces have to be so that there is zero net force acting on each object (and zero net torque, too, but that's the topic of Section 2.2). Because a force is a vector, this goal involves breaking the force up into its components. You can pick Cartesian coordinates, polar coordinates, or another set. It is usually clear from the problem which system will make your calculations easiest. Once you pick a system, you simply have to demand that the total external force in each direction is zero.

There are many different types of forces in the world, most of which are large-scale effects of complicated things going on at smaller scales. For example, the tension in a rope comes from the chemical bonds that hold the molecules in the rope together, and these chemical forces are electrical forces. In doing a mechanics problem involving a rope, there is certainly no need to analyze all the details of the forces taking place at the molecular scale. You just call the force in the rope a "tension" and get on with the problem. Four types of forces come up repeatedly:

Tension

Tension is the general name for a force that a rope, stick, etc., exerts when it is pulled on. Every piece of the rope feels a tension force in both directions, except the end points, which feel a tension on one side and a force on the other side from whatever object is attached to the end. In some cases, the tension may vary along the rope. The "Rope wrapped around a pole" example at the end of this section is a good illustration of this. In other cases, the tension must be the same everywhere. For example, in a hanging massless rope, or in a massless rope hanging over a frictionless pulley, the tension must be the same at all points, because otherwise there would be a net force on at least some part of the rope, and then $F = ma$ would yield an infinite acceleration for this (massless) piece.

Normal force

This is the force perpendicular to a surface that the surface applies to an object. The total force applied by a surface is usually a combination of the normal force and the friction force (see below). But for frictionless surfaces such as greasy ones or ice, only the normal force exists. The normal force comes about because the surface actually compresses a tiny bit and acts like a very rigid spring. The surface gets squashed until the restoring force equals the force necessary to keep the object from squashing in any more.

For the most part, the only difference between a "tension" and a "normal force" is the direction of the force. Both situations can be modeled by a spring. In the case of a tension, the spring (a rope, a stick, or whatever) is stretched, and the force on the given object is directed toward the spring. In the case of a normal force, the spring is compressed, and the force on the given object is directed away from the spring. Things like sticks can provide both normal forces and tensions. But a rope, for example, has a hard time providing a normal force. In practice, in the case of elongated objects such as sticks, a compressive force is usually called a "compressive tension," or a "negative tension," instead of a normal force. So by these definitions, a tension can point either way. At any rate, it's just semantics. If you use any of these descriptions for a compressed stick, people will know what you mean.

Friction

Friction is the force parallel to a surface that a surface applies to an object. Some surfaces, such as sandpaper, have a great deal of friction. Some, such as greasy ones, have essentially no friction. There are two types of friction, called "kinetic" friction and "static" friction. Kinetic friction (which we won't cover in this chapter) deals with two objects moving relative to each other. It is usually a good approximation to say that the kinetic friction between two objects is proportional to the normal force between them. The constant of proportionality is called μ_k (the "coefficient of kinetic friction"), where μ_k depends on the two surfaces involved. Thus, $F = \mu_k N$, where N is the normal force. The direction of the force is opposite to the motion.

Static friction deals with two objects at rest relative to each other. In the static case, we have $F \le \mu_s N$ (where μ_s is the "coefficient of static friction"). Note the inequality sign. All we can say prior to solving a problem is that the static friction force has a *maximum* value equal to $F_{max} = \mu_s N$. In a given problem, it is most likely less than this. For example, if a block of large mass M sits on a surface with coefficient of friction μ_s, and you give the block a tiny push to the right (tiny enough so that it doesn't move), then the friction force is of course not equal to $\mu_s N = \mu_s Mg$ to the left. Such a force would send the block sailing off to the left. The true friction force is simply equal and opposite to the tiny force you apply. What the coefficient μ_s tells us is that if you apply a force larger than $\mu_s Mg$ (the maximum friction force on a horizontal table), then the block will end up moving to the right.

Gravity

Consider two point objects, with masses M and m, separated by a distance R. Newton's gravitational force law says that the force between these objects is attractive and has magnitude $F = GMm/R^2$, where $G = 6.67 \cdot 10^{-11}\ \text{m}^3/(\text{kg s}^2)$. As we'll show in Chapter 5, the same law also applies to spheres of nonzero size. That is, a sphere may be treated like a point mass located at its center. Therefore, an object on the surface of the earth feels a gravitational force equal to

$$F = m\left(\frac{GM}{R^2}\right) \equiv mg, \tag{2.1}$$

where M is the mass of the earth, and R is its radius. This equation defines g. Plugging in the numerical values, we obtain $g \approx 9.8\ \text{m/s}^2$, as you can check. Every object on the surface of the earth feels a force of mg downward (g varies slightly over the surface of the earth, but let's ignore this). If the object is not accelerating, then there must be other forces present (normal forces, etc.) to make the total force be equal to zero.

Another common force is the Hooke's-law spring force, $F = -kx$. But we'll postpone the discussion of springs until Chapter 4, where we'll spend a whole chapter on them in depth.

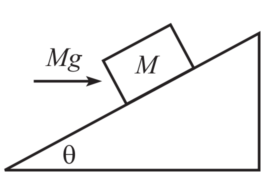

Fig. 2.1

Example (Block on a plane): A block of mass M rests on a fixed plane inclined at an angle θ. You apply a horizontal force of Mg on the block, as shown in Fig. 2.1. Assume that the friction force between the block and the plane is large enough to keep the block at rest. What are the normal and friction forces (call them N and F_f) that the plane exerts on the block? If the coefficient of static friction is μ, for what range of angles θ will the block in fact remain at rest?

Solution: Let's break the forces up into components parallel and perpendicular to the plane. (The horizontal and vertical components would also work, but the calculation would be a little longer.) The forces are N, F_{f}, the applied Mg, and the weight Mg, as shown in Fig. 2.2. Balancing the forces parallel and perpendicular to the plane gives, respectively (with upward along the plane taken to be positive),

$$
\begin{aligned}
F_{\text{f}} &= Mg\sin\theta - Mg\cos\theta, \\
N &= Mg\cos\theta + Mg\sin\theta.
\end{aligned}
\tag{2.2}
$$

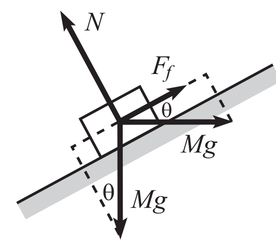

Fig. 2.2

INTERMEDIATE REMARKS:

1. If $\tan\theta > 1$, then F_{f} is positive (that is, it points up the plane). And if $\tan\theta < 1$, then F_{f} is negative (that is, it points down the plane). There is no need to worry about which way it points when drawing the diagram. Just pick a direction to be positive, and if F_{f} comes out to be negative (as it does in the figure above, because $\theta < 45°$), then it actually points in the other direction.
2. F_{f} ranges from $-Mg$ to Mg as θ ranges from 0 to $\pi/2$ (convince yourself that these limiting values make sense). As an exercise, you can show that N is maximum when $\tan\theta = 1$, in which case $N = \sqrt{2}Mg$ and $F_{\text{f}} = 0$.
3. The $\sin\theta$ and $\cos\theta$ factors in Eq. (2.2) follow from the angles θ drawn in Fig. 2.2. However, when solving problems like this one, it's easy to make a mistake in the geometry and then label an angle as θ when it really should be $90° - \theta$. So two pieces of advice: (1) Never draw an angle close to 45° in a figure, because if you do, you won't be able to tell the θ angles from the $90° - \theta$ ones. (2) Always check your results by letting θ go to 0 or 90° (in other words, does virtually all of a force, or virtually none of it, act in a certain direction when the plane is, say, horizontal). Once you do this a few times, you'll realize that you probably don't even need to work out the geometry in the first place. Since you know that any given component is going to involve either $\sin\theta$ or $\cos\theta$, you can just pick the one that works correctly in a certain limit. ♣

The coefficient μ tells us that $|F_{\text{f}}| \le \mu N$. Using Eq. (2.2), this inequality becomes

$$
Mg|\sin\theta - \cos\theta| \le \mu Mg(\cos\theta + \sin\theta). \tag{2.3}
$$

The absolute value here signifies that we must consider two cases:

- If $\tan\theta \ge 1$, then Eq. (2.3) becomes

$$
\sin\theta - \cos\theta \le \mu(\cos\theta + \sin\theta) \quad \Longrightarrow \quad \tan\theta \le \frac{1+\mu}{1-\mu}. \tag{2.4}
$$

  We divided by $1 - \mu$, so this inequality is valid only if $\mu < 1$. But if $\mu \ge 1$, we see from the first inequality here that any value of θ (subject to our assumption, $\tan\theta \ge 1$) works.
- If $\tan\theta \le 1$, then Eq. (2.3) becomes

$$
-\sin\theta + \cos\theta \le \mu(\cos\theta + \sin\theta) \quad \Longrightarrow \quad \tan\theta \ge \frac{1-\mu}{1+\mu}. \tag{2.5}
$$

Putting these two ranges for θ together, we have

$$\frac{1-\mu}{1+\mu} \le \tan\theta \le \frac{1+\mu}{1-\mu}. \tag{2.6}$$

REMARKS: For very small μ, these bounds both approach 1, which means that θ must be very close to 45°. This makes sense. If there is very little friction, then the components along the plane of the horizontal and vertical Mg forces must nearly cancel; hence, $\theta \approx 45°$. A special value for μ is 1, because from Eq. (2.6), we see that $\mu = 1$ is the cutoff value that allows θ to reach both 0 and $\pi/2$. If $\mu \ge 1$, then any tilt of the plane is allowed. We've been assuming throughout this example that $0 \le \theta \le \pi/2$. The task of Exercise 2.20 is to deal with the case where $\theta > \pi/2$, where the block is under an overhang. ♣

Let's now do an example involving a rope in which the tension varies with position. We'll need to consider differential pieces of the rope to solve this problem.

Example (Rope wrapped around a pole): A rope wraps an angle θ around a pole. You grab one end and pull with a tension T_0. The other end is attached to a large object, say, a boat. If the coefficient of static friction between the rope and the pole is μ, what is the largest force the rope can exert on the boat, if the rope is not to slip around the pole?

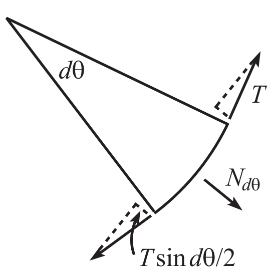

Fig. 2.3

Solution: Consider a small piece of the rope that subtends an angle $d\theta$. Let the tension in this piece be T (which varies slightly over the small length). As shown in Fig. 2.3, the pole exerts a small outward normal force, $N_{d\theta}$, on the piece. This normal force exists to balance the "inward" components of the tensions at the ends. These inward components have magnitude $T\sin(d\theta/2)$.[1] Therefore, $N_{d\theta} = 2T\sin(d\theta/2)$. The small-angle approximation, $\sin x \approx x$, allows us to write this as $N_{d\theta} = T\,d\theta$.

The friction force on the little piece of rope satisfies $F_{d\theta} \le \mu N_{d\theta} = \mu T\,d\theta$. This friction force is what gives rise to the difference in tension between the two ends of the piece. In other words, the tension, as a function of θ, satisfies

$$\begin{aligned} T(\theta + d\theta) &\le T(\theta) + \mu T\,d\theta \\ \implies \quad dT &\le \mu T\,d\theta \\ \implies \quad \int \frac{dT}{T} &\le \int \mu\,d\theta \\ \implies \quad \ln T &\le \mu\theta + C \\ \implies \quad T &\le T_0 e^{\mu\theta}, \end{aligned} \tag{2.7}$$

[1] One of them actually has magnitude $(T + dT)\sin(d\theta/2)$, where dT is the increase in tension along the small piece. But the extra term this produces, $(dT)\sin(d\theta/2)$, is a second-order small quantity, so it can be ignored.

where we have used the fact that $T = T_0$ when $\theta = 0$. The exponential behavior here is quite strong (as exponential behaviors tend to be). If we let $\mu = 1$, then just a quarter turn around the pole produces a factor of $e^{\pi/2} \approx 5$. One full revolution yields a factor of $e^{2\pi} \approx 530$, and two full revolutions yield a factor of $e^{4\pi} \approx 300\,000$. Needless to say, the limiting factor in such a case is not your strength, but rather the structural integrity of the pole around which the rope winds.

2.2 Balancing torques

In addition to balancing forces in a statics problem, we must also balance torques. We'll have much more to say about torque in Chapters 8 and 9, but we'll need one important fact here. Consider the situation in Fig. 2.4, where three forces are applied perpendicular to a stick, which is assumed to remain motionless. F_1 and F_2 are the forces at the ends, and F_3 is the force in the interior. We have, of course, $F_3 = F_1 + F_2$, because the stick is at rest. But we also have the following relation:

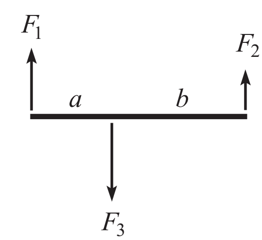

Fig. 2.4

Claim 2.1 *If the system is motionless, then $F_3 a = F_2(a + b)$. In other words, the torques (force times distance) around the left end cancel.*[2] *And you can show that they cancel around any other point, too.*

We'll prove this claim in Chapter 8 by using angular momentum, but let's give a short proof here.

Proof: We'll make one reasonable assumption, namely, that the correct relationship between the forces and distances is of the form,

$$F_3 f(a) = F_2 f(a + b), \tag{2.8}$$

where $f(x)$ is a function to be determined.[3] Applying this assumption with the roles of "left" and "right" reversed in Fig. 2.4 gives

$$F_3 f(b) = F_1 f(a + b). \tag{2.9}$$

Adding Eqs. (2.8) and (2.9), and using $F_3 = F_1 + F_2$, yields

$$f(a) + f(b) = f(a + b). \tag{2.10}$$

This equation implies that $f(rx) = rf(x)$ for any x and for any rational number r, as you can show (see Exercise 2.28). Therefore, assuming $f(x)$ is continuous,

[2] Another proof of this claim is given in Problem 2.11.

[3] What we're doing here is simply assuming linearity in F. That is, two forces of F applied at a point should be the same as a force of $2F$ applied at that point. You can't really argue with that.

it must be a linear function, $f(x) = Ax$, as we wanted to show. The constant A is irrelevant, because it cancels in Eq. (2.8). ■

Note that dividing Eq. (2.8) by Eq. (2.9) gives $F_1 f(a) = F_2 f(b)$, and hence $F_1 a = F_2 b$, which says that the torques cancel around the point where F_3 is applied. You can show that the torques cancel around any arbitrary pivot point. When adding up all the torques in a given physical setup, it is of course required that you use the same pivot point when calculating each torque.

In the case where the forces aren't perpendicular to the stick, the above claim applies to the components of the forces perpendicular to the stick. This makes sense, because the components parallel to the stick have no effect on the rotation of the stick around the pivot point. Therefore, referring to Figs. 2.5 and 2.6, the equality of the torques can be written as

$$F_a a \sin\theta_a = F_b b \sin\theta_b. \tag{2.11}$$

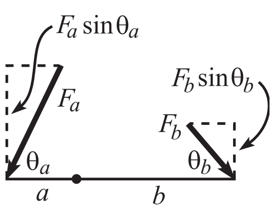

Fig. 2.5

This equation can be viewed in two ways:

- $(F_a \sin\theta_a)a = (F_b \sin\theta_b)b$. In other words, we effectively have smaller forces acting on the given "lever arms," as shown in Fig. 2.5.
- $F_a(a \sin\theta_a) = F_b(b \sin\theta_b)$. In other words, we effectively have the given forces acting on smaller "lever arms," as shown in Fig. 2.6.

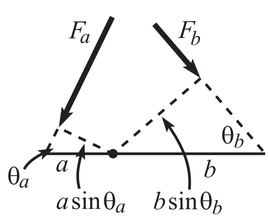

Fig. 2.6

Claim 2.1 shows that even if you apply only a tiny force, you can balance the torque due to a very large force, provided that you make your lever arm sufficiently long. This fact led a well-known mathematician of long ago to claim that he could move the earth if given a long enough lever arm.

> One morning while eating my Wheaties,
> I felt the earth move 'neath my feeties.
> The cause for alarm
> Was a long lever arm,
> At the end of which grinned Archimedes!

One handy fact that comes up often is that the gravitational torque on a stick of mass M is the same as the gravitational torque due to a point-mass M located at the center of the stick. The truth of this statement relies on the fact that torque is a linear function of the distance to the pivot point (see Exercise 2.27). More generally, the gravitational torque on an object of mass M may be treated simply as the gravitational torque due to a force Mg located at the center of mass.

We'll talk more about torque in Chapters 8 and 9, but for now we'll just use the fact that in a statics problem the torques around any given point must balance.

Example (Leaning ladder): A ladder leans against a frictionless wall. If the coefficient of friction with the ground is μ, what is the smallest angle the ladder can make with the ground and not slip?

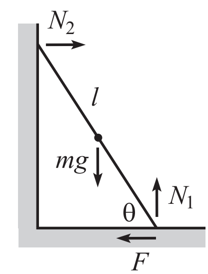

Fig. 2.7

Solution: Let the ladder have mass m and length ℓ. As shown in Fig. 2.7, we have three unknown forces: the friction force F, and the normal forces N_1 and N_2. And to solve for these three forces we fortunately have three equations: $\Sigma F_{\text{vert}} = 0$, $\Sigma F_{\text{horiz}} = 0$, and $\Sigma\tau = 0$ (τ is the standard symbol for torque). Looking at the vertical forces, we see that $N_1 = mg$. And then looking at the horizontal forces, we see that $N_2 = F$. So we have quickly reduced the unknowns from three to one.

We will now use $\Sigma\tau = 0$ to find N_2 (or F). But first we must pick the "pivot" point around which we will calculate the torques. Any stationary point will work fine, but certain choices make the calculation easier than others. The best choice for the pivot is generally the point at which the most forces act, because then the $\Sigma\tau = 0$ equation will have the smallest number of terms in it (because a force provides no torque around the point where it acts, since the lever arm is zero). In this problem, there are two forces acting at the bottom end of the ladder, so this is the point we'll choose for the pivot (but you should verify that other choices for the pivot, for example, the middle or top of the ladder, give the same result). Balancing the torques due to gravity and N_2, we have

$$N_2\ell\sin\theta = mg(\ell/2)\cos\theta \quad\Longrightarrow\quad N_2 = \frac{mg}{2\tan\theta}. \tag{2.12}$$

This is also the value of the friction force F. The condition $F \le \mu N_1 = \mu mg$ therefore becomes

$$\frac{mg}{2\tan\theta} \le \mu mg \quad\Longrightarrow\quad \tan\theta \ge \frac{1}{2\mu}. \tag{2.13}$$

Remarks: Note that the total force exerted on the ladder by the floor points up at an angle given by $\tan\beta = N_1/F = (mg)/(mg/2\tan\theta) = 2\tan\theta$. We see that this force does *not* point along the ladder. There is simply no reason why it should. But there *is* a nice reason why it should point upward with twice the slope of the ladder. This is the direction that causes the lines of the three forces on the ladder to be concurrent (that is, pass through a common point), as shown in Fig. 2.8. This concurrency is a neat little theorem for statics problems involving three forces. The proof is simple. If the three lines weren't concurrent, then one force would produce a nonzero torque around the intersection point of the other two lines of force.[4]

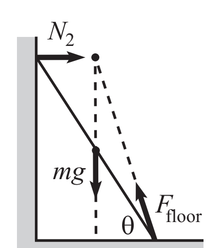

Fig. 2.8

This theorem provides a quick way to solve the ladder problem in the more general case where the center of mass is a fraction f of the way up. In this case, the concurrency theorem tells us that the slope of the total force from the floor is $(1/f)\tan\theta$, consistent with the $f = 1/2$ result from above. The vertical component is still mg, so the horizontal (friction) component is now $fmg/\tan\theta$. Demanding that this be less than or equal to μmg gives $\tan\theta \ge f/\mu$, consistent with the $f = 1/2$ result. Since this result depends

[4] The one exception to this reasoning is where no two of the lines intersect, that is, where all three lines are parallel. Equilibrium is certainly possible in such a scenario, as we saw in Claim 2.1. But you can hang on to the concurrency theorem in this case if you consider the parallel lines to meet at infinity.

only on the location of the center of mass, and not on the exact distribution of mass, a corollary is that if you climb up a ladder (resting on a frictionless wall), your presence makes the ladder more likely to slip if you are above the center of mass (because you have raised the center of mass of the entire system and thus increased f), and less likely if you are below. ♣

The examples we've done in this chapter have consisted of only one object. But many problems involve more than one object (as you'll find in the problems and exercises for this chapter), and there's one additional fact you'll often need to invoke for these, namely Newton's third law. This states that the force that object A exerts on object B is equal and opposite to the force that B exerts on A (we'll talk more about Newton's laws in Chapter 3). So if you want to find, say, the normal force between two objects, you might be able to figure it out by looking at forces and torques on either object, depending on how much you already know about the other forces acting on each. Once you've found the force by dealing with, say, object A, you can then use the equal and opposite force to help figure out things about B. Depending on the problem, one object is often more useful than the other to use first.

Note, however, that if you pick your subsystem (on which you're going to consider forces and torques) to include both A and B, then this won't tell you anything at all about the normal force (or friction) between them. This is true because the normal force is an *internal* force between the objects (when considered together as a system), whereas only *external* forces are relevant in calculating the total force and torque on the system (because all the internal forces cancel in pairs, by Newton's third law). The only way to determine a given force is to deal with it as an external force on some subsystem(s).

Statics problems often involve a number of decisions. If there are various parts to the system, then you must decide which subsystems you want to balance the external forces and torques on. And furthermore, you must decide which point to use as the origin for calculating the torques. There are invariably many choices that will give you the information you need, but some will make your calculations much cleaner than others (Exercise 2.35 is a good example of this). The only way to know how to choose wisely is to start solving problems, so you may as well tackle some . . .

2.3 Problems

Section 2.1: Balancing forces

2.1. **Hanging rope**

A rope with length L and mass density per unit length ρ is suspended vertically from one end. Find the tension as a function of height along the rope.