Lecture-2 inclusion - exclusion

of onto mapping from an m sets to an n-set

$A = [m]$ $\quad\quad$ $B = [n]$

m $\quad\quad$ n

X = set of all functions from A to B.

$|X| = n^m = |B|^{|A|}$, so $B^A = \{$set of all functions from A to $B\}$

$A_1 = \{ f : A \to B \mid$ range f, does not contains $1\}$

$\vdots$

$A_n = \{ f : A \to B \mid$ range f, does not contains $n\}$
i.e $n \notin$ range (f).

$$\left| X - \bigcup_{j=1}^{m} A_j \right|$$

so $A_1, \ldots, A_n$ are set of all non onto functions

so,

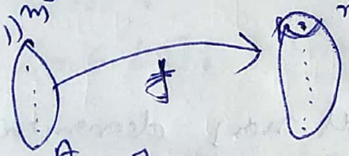
$\left| X - \bigcup_{i=1}^{m} A_i \right|$ = numbers of onto functions from A to B

$$\sum_{I \subseteq [n]} (-1)^{|I|} |A_I|, \text{ where } A_I = \bigcap_{i \in I} A_i$$

$$= |X| - \sum |A_i| + \sum |A_i \cap A_j| - \ldots + \ldots$$

$|X|$ = set of all functions from A to B $= n^m$

$|A_1| = (n-1)^m$



$A\ [m]$ $\quad\quad$ $B\ [n]$

not in range.

so $|A_i| = (n-1)^m$

so

$\Sigma A_i = \binom{n}{1} (n-1)^m$

so

$|A_1 \cap A_2| = (n-2)^m$, $\quad |A_i \cap A_j| = (n-2)^m$

$\Sigma |A_i \cap A_j| = \binom{n}{2} (n-2)^m$

$\therefore$ # of onto mapping from an m set to an n set

is

$$\sum_{i=0}^{m} \quad n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + \cdots$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$$

Put $m=n$ in this we must get $n!$

⁂ Derangement of an n-set, n letters & n envelopes

d_n: number of derangements of an n-set say $[n]$.

π : Permutation of $[n]$

$\pi(i) \neq i \quad \forall\ i \in [n]$

$n=3$ (n) 1, 2, 3

π $\pi(n)$ 2 3 1 so $d_3 = 2$

3 1 2

$$\boxed{d_n = (n-1)(d_{n-1} + d_{n-2})}$$ (we can prove)

table of derangements.

n	1	2	3	4	5	6	7	8
d_n	0	1	2	9	44	265	1854	14833

X = set of all permutations of $[n]$

$|X| = n!$

$A_1 = \{\pi : \pi(1) = 1\}$

$\vdots$

$A_n = \{\pi : \pi(n) = n\}$

$\bigcup_{i=1}^{n} A_i$ represent all non derangement permutation.

$|A_1| = (n-1)! \quad \cdots\cdots \quad |A_i| = (n-1)!$

$\Rightarrow \sum |A_i| = \binom{n}{1}(n-1)! = n!$

$|A_1 \cap A_2| = (n-2)!$ so $\cdots$ $|A_i \cap A_j| = (n-2)!$ $i \neq j$

$\Rightarrow \sum |A_i \cap A_j| = \binom{n}{2}(n-2)!$

from PIE, $\sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$

$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$

$= n! \left[\sum (-1)^i \frac{1}{i!}\right]$

$= n!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots\cdots\right)$

so $\frac{d_n}{n!} = \sum_{i=0}^{n} \frac{(-1)^i}{i!} \rightarrow e^{-1}$ as $n \to \infty$

* let $[n] = \{1, 2, 3 \ldots n\}$

$S_n = \{a \in [n] : \gcd(a, n) = 1\}$

e.g. $S_{12} = \{1, 5, 7, 11\}$

$|S_{12}| = 4$

$|S_n| = \phi(n)$, ϕ : Euler's totient functions phi,

table of $\phi(n)$

n	1,	2	3	4	5	6	7	8	9	10
$\phi(n)$	1	1	2	2	4	2	6	4	6	4

$X = [n] = \{1, 2, 3 \ldots n\}$.

let $P_1, P_2, P_3 \ldots\ldots P_k$ be the prime factor of n.

$A_1 = \{m \in [n] : P_1 | m\}$ divides symbol.

$\vdots$

$A_k = \{m \in [n] ; P_k | m\}$

$|A_1| = \frac{n}{P_1} \quad \ldots \quad A_k = \frac{n}{P_k}$

$|A_1 \cap A_2| = \frac{n}{P_1 P_2} \quad \ldots \quad |A_i \cap A_j| = \frac{n}{P_i P_j}$

$|A_1 \cap A_2 \cap A_3| = \frac{n}{P_1 P_2 P_3}$,

$|A_i \cap A_j \cap A_l| = \frac{n}{P_i P_j P_l}$

from PIE :-

$$\phi(n) = |X| - \sum|A_i| + \sum|A_i \cap A_j| - + \cdots \cdots$$

$$= n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \cdots \frac{n}{p_k}\right) + \left(\frac{n}{p_1 p_2} + \cdots \frac{n}{p_{k-1} p_k}\right)$$

$$- \cdots + \cdots$$

$$= \quad + (-1)^k \frac{n}{p_1 \cdots p_k}$$

$$= n\left(1 - \sum \frac{1}{p_i} + \sum \frac{1}{p_i p_j} - \cdots + (-1)^k \frac{1}{p_1 \cdots p_k}\right)$$

$$= n(1 - 1/p_1)(1 - 1/p_2) \cdots (1 - 1/p_k)$$

$$\phi(n) = (p_1 - 1)(p_2 - 1) \cdots \cdots (p_k - 1)$$

$$\Rightarrow \phi(n) = n \prod_{p|n} (1 - 1/p)$$

$$\phi(1) = 1$$