PYQ5

Question. Let ABC be a triangle with incentre I and inradius r. Let D, E and F be the feet of the perpendiculars from I to the sides BC, CA and AB respectively. If r_1, r_2, r_3 are the radii of circles inscribed in the quadrilaterals AFIE, BDIF and CEID respectively, prove that

$$\frac{r_1}{r-r_1} + \frac{r_2}{r-r_2} + \frac{r_3}{r-r_3} = \frac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}$$

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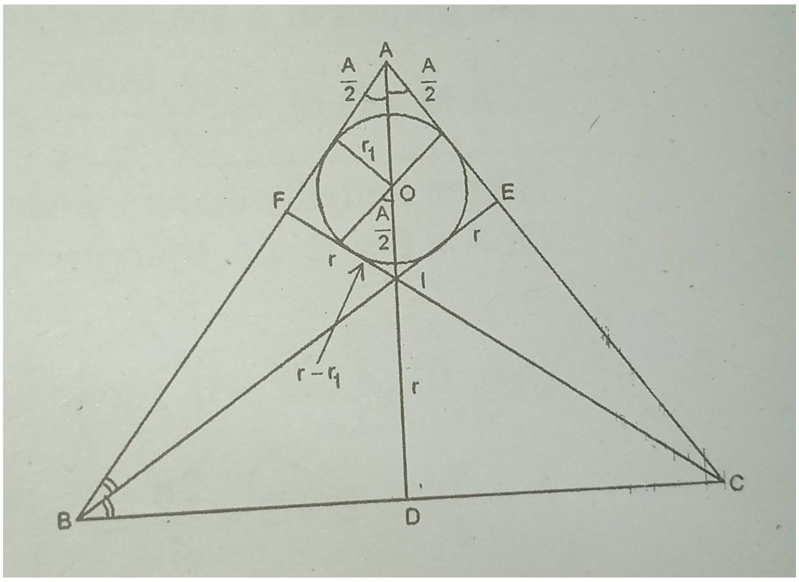
Solution.



From figure, it is clear that

$$\frac{r-r_1}{r_1} = \tan\frac{A}{2}$$

$$\Rightarrow \quad \frac{r_1}{r-r_1} = \cot\frac{A}{2} \quad \text{—— (i)}$$

Similarly, $\frac{r_2}{\quad} = \cot\frac{B}{\quad}$ —— (ii)

Similarly, $$\frac{r_2}{r-r_2} = \cot\frac{B}{2} \quad \text{—— (ii)}$$

and $$\frac{r_3}{r-r_3} = \cot\frac{C}{2} \quad \text{—— (iii)}$$

Now, $$A + B + C = \pi$$

$$\Rightarrow \quad \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right) = \tan\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} - \tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}{1 - \tan\frac{A}{2}\cdot\tan\frac{B}{2} - \tan\frac{B}{2}\cdot\tan\frac{C}{2} - \tan\frac{C}{2}\cdot\tan\frac{A}{2}} = \infty$$

$$\Rightarrow \quad \tan\frac{A}{2}\cdot\tan\frac{B}{2} + \tan\frac{B}{2}\cdot\tan\frac{C}{2} + \tan\frac{C}{2}\cdot\tan\frac{A}{2} = 1$$

Dividing both sides by $\tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}$: →

$$\Rightarrow \quad \frac{1}{\tan\frac{C}{2}} + \frac{1}{\tan\frac{A}{2}} + \frac{1}{\tan\frac{B}{2}} = \frac{1}{\tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}$$

$$\Rightarrow \quad \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$$

$$\Rightarrow \boxed{\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}}$$

Proved.