Question. Internal bisector of $\angle A$ of ΔABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and the side AB at F. If a, b, c represent sides of ΔABC then:→

[Multiple correct]

[IIT 2006, 05]

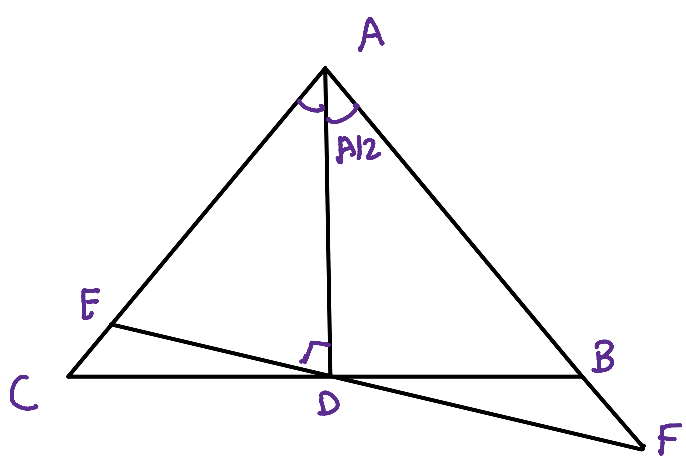
(A) AE is HM of b and c

(B) $AD = \frac{2bc}{b+c} \cos\frac{A}{2}$

(C) $EF = \frac{4bc}{b+c} \cdot \sin\frac{A}{2}$

(D) the ΔAEF is isosceles

Solution.



By Angle Bisector Rule, we know that:→

$$\frac{AC}{AB} = \frac{CD}{BD} = \frac{m}{n} \text{ (let)}$$

Now, using Cosine rule in ΔACD,

$$CD^2 = AC^2 + AD^2 - 2AC \cdot AD \cdot \cos\frac{A}{2} \quad \text{—①}$$

Similarly, in ΔABD,

$$BD^2 = AB^2 + AD^2 - 2AB \cdot AD \cdot \cos\frac{A}{2} \quad \text{— (11)}$$

$$\therefore \quad \frac{CD^2}{BD^2} = \frac{AC^2 + AD^2 - 2AC \cdot AD \cdot \cos A/2}{AB^2 + AD^2 - 2AB \cdot AD \cdot \cos A/2}$$

$$\Rightarrow \quad \frac{AC^2}{AB^2} = \frac{AC^2 + AD^2 - 2AC \cdot AD \cdot \cos A/2}{AB^2 + AD^2 - 2AB \cdot AD \cdot \cos A/2}$$

$$\Rightarrow \quad \cancel{AC^2 \cdot AB^2} + AC^2 \cdot AD^2 - 2AB \cdot AD \cdot AC^2 \cdot \cos A/2$$
$$= \cancel{AB^2 \cdot AC^2} + AD^2 \cdot AB^2 - 2AC \cdot AD \cdot AB^2 \cdot \cos A/2$$

$$\Rightarrow \quad AD^{\cancel{2}}\left[\cancel{(AC - AB)}\,(AC + AB)\right] = 2AC \cdot AB \cdot A\cancel{D} \cdot \cos\frac{A}{2}\left\{\cancel{(AC - AB)}\right\}$$

$$\Rightarrow \quad AD = \frac{2AC \cdot AB}{AB + AC} \cdot \cos\frac{A}{2}$$

$$\Rightarrow \quad \boxed{AD = \frac{2bc}{b+c} \cdot \cos\frac{A}{2}}$$ Option (B)

Now, in ΔADE,

$$\sec\frac{A}{2} = \frac{AE}{AD}$$

$$\Rightarrow \quad AE = AD \sec\frac{A}{2}$$

$$\boxed{AE = \frac{2bc}{b+c}}$$

$b+c$

$\therefore$ AE is the HM of b and c. Ans: (A)

Now,

$$EF = ED + DF$$

$$= AD\tan\frac{A}{2} + AD\tan\frac{A}{2}$$

$$= 2AD\tan\frac{A}{2}$$

$$\boxed{EF = 4\frac{bc}{b+c}\sin\frac{A}{2}}$$ Ans (C)

Now, $DE = DF = AD\tan\frac{A}{2}$

$\therefore$ $AF = AE = AD\sec\frac{A}{2}$

$\therefore$ ΔAEF is isosceles. Ans: (D)

$\therefore$ Correct options are:→ (A), (B), (C), (D). Ans.