LIMITS

Let $\lim_{x \to a} = A$. It would mean that when we approach the point $x = a$ from the values which are just greater than or smaller than $x = a$.

Indeterminate forms:

There are seven indeterminant forms.

(I) $\frac{0}{0}$, $N^r \to 0$, $D^r \to 0$

(II) $\frac{\infty}{\infty}$, $N^r \to$, $D^r \to 0$

(III) $0 \cdot \infty$, one factor $\to 0$ and other factor $\to \infty$

(IV) $\infty - \infty$, each $\to 0$

(V) 1^{∞}, base $\to 1$, power $\to \infty$

(VI) 0^0, base $\to 0$, power $\to 0$

(VII) ∞^0, base $\to \infty$, power $\to 0$

Properties of infinity :→

(I) $\infty \pm c = \infty$

(II) $\infty + \infty = \infty$

(III) $\infty \times \infty = \infty$

(IV) $\infty^{\infty} = \infty$

(V) (I) $0 \times \infty = 0$
$\uparrow$
exact zero

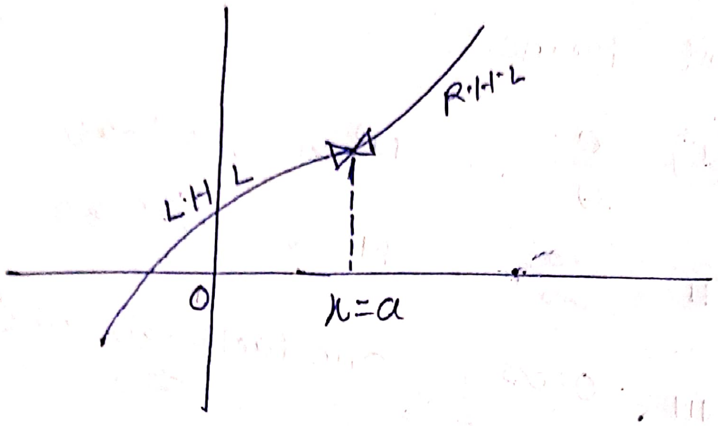
(II) $0 \times \infty$ = undeterminant
$\uparrow$
tends to zero.

Existance of limit :→

1. Limit does exist :→

let $\lim_{x \to a} f(x) = A$

If x tends to a then $f(x)$ → to A



$L \cdot H \cdot L = \lim_{x \to a^-} f(x)$, when a^- = less than a

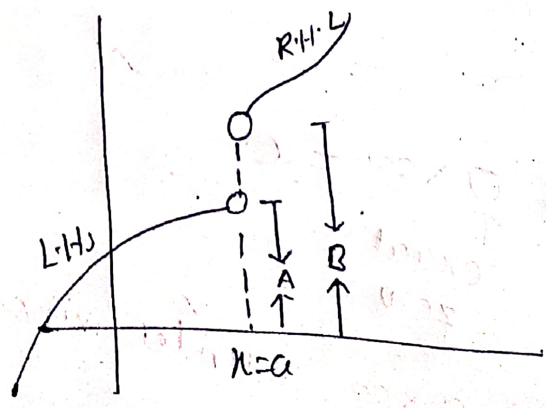
$R \cdot H \cdot L = \lim_{x \to a^+} f(x)$ when a^+ = greater than a

If L·H·L = R·H·L

then limit does exist at $x=a$.

2. Limit does not exist :→

(I) In graphical



Here L·H·L = A
R·H·L = B

L·H·L ≠ R·H·L (Hence does not exist)

(II) L·H·L or R·H·L does not exist

(III) L·H·L ≠ R·H·L by

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

(IV) when the function shown interacting behaviour. ~~in the never rw~~

Ex. $\lim_{x \to 0} \sin 1/x$ (does not exist)

$\lim_{x \to 0} 5^{1/x}$ (does not exist)

Answers of limits

(I) R (Real)
(II) $+\infty$
(III) $-\infty$
(IV) Does not exist

Note :→ In case of indeterminant forms all the value given here are approximatelly not adjacent value.

Basic law of limits:→

(I) $\lim_{x\to a}\{f(x)\pm g(x)\} = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(II) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{l_1}{l_2}$

$l_2 \neq 0$

(III) $\lim_{x\to a} f\{g(x)\} = f\{\lim_{x\to a} g(x)\}$

Some important expansions:→

(I) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\cdots$

(II) $a^x = 1 + \dfrac{\log a \cdot x}{1!} + \dfrac{(\log a)^2 \cdot x^2}{2!} + \dfrac{(\log a)^3 \cdot x^3}{3!} + \cdots$

$a \in R^+$

(III) $(1+x)^n = 1 + \dfrac{nx}{1!} + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \cdots\cdots$

$n \in R$ and $|x| < 1$

(IV) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots\cdots$

(V) $(1+x)^{1/x} = e\left[1 - \dfrac{x}{2} + \dfrac{11x^2}{24} + \cdots\right]$

(VI) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

(VII) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

VIII $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$

Note Given expansion is proved by Macau Maclaurin's series.

Some important results

1. Theorem of :-

$$\boxed{\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}}$$

where n is rational number.

Proof:-

let $x = a+h$

If $x \to a$ then $h \to 0$

Now,

$$L.H.S = \lim_{h\to 0} \frac{(a+h)^n - a^n}{\cancel{a}+h-\cancel{a}}$$

$$= \lim_{h\to 0} \frac{a^n\left[\left(1+\frac{h}{a}\right)^n - 1\right]}{h}$$

$$= \lim_{h\to 0} \frac{a^n\left[\cancel{1}+\frac{n\cdot\left(\frac{h}{a}\right)}{1!} + \frac{n(n-1)\left(\frac{h}{a}\right)^2}{2!} + \cdots\right]}{h}$$

$$= \lim_{h\to 0} \frac{a^n \cancel{h}\left[\frac{n}{a} + \frac{n(n-1)\cdot\frac{h}{a^2}}{2!} + \cdots\right]}{\cancel{h}}$$

$$= a^n \cdot \frac{n}{a}$$

$$= na^{n-1}$$

Type-I

problem based on limits of algebraic function:-

(Q) evaluate

(I) $\lim_{x \to 5} (x^3 + 6)$ (not) indeterminate fn

(II) $\lim_{x \to 0} [x^2 + 2x + 1]$

(III) $\lim_{x \to 1} \frac{x^2 + 3x + 2}{x^2 + 1}$ not indeterminate fn

Q. Evaluate

$$\lim_{x \to a} \frac{x^{5/2} - a^{5/2}}{\sqrt{x} - \sqrt{a}}$$

Soln:→ $\frac{0}{0}$ form

$$\lim_{x \to a} \frac{\frac{x^{5/2} - a^{5/2}}{x - a}}{\frac{x^{1/2} - (a)^{1/2}}{x - a}}$$

$$= \frac{\frac{5}{2}(a)^{\frac{5}{2} - 1}}{\frac{1}{2}(a)^{\frac{1}{2} - 1}}$$

$$= 5(a)^{\frac{3}{2} + \frac{1}{2}}$$

$$= 5a^2. \text{ Ans}$$

Q Evaluate

$$\text{limit}_{x \to 0} \frac{(1+x)^5 - 1}{3x + 5x^2} \quad \frac{0}{0} \text{ form.}$$

$$= \lim_{x \to 0} \frac{\frac{(1+x)^5 - 1^5}{(1+x) - 1} \times \cancel{(1+x-1)}}{\cancel{x}(3 + 5x)}$$

$$= \frac{(5)\,1^4}{3} = \frac{5}{3}.$$

2nd method

$$\lim_{x \to 0} \frac{(\cancel{1} + 5x + 10x^2 + 10x^3 + 5x^4 + x^5) - \cancel{1}}{x(3+5x)}$$

$= \frac{5}{3}$ Ans

Q Evaluate

(I) $\lim_{x \to 0} \frac{\sqrt{1+4x} - \sqrt{1-3x}}{x}$

(III) $\lim_{x \to 0} \frac{\sqrt{1+x^2} - \sqrt{1+x}}{\sqrt{x^2+1} + \sqrt{x+1}}$

not $\frac{0}{0}$ form

(II) $\lim_{x \to 0} \frac{\sqrt{x+a} - \sqrt{a}}{x\sqrt{a(a+x)}}$

(IV) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1+x}}{\sqrt{1+x^2} - \sqrt{1+x}}$

Double rationalisation

Sol^n:-

(I) $\lim_{x \to 0} \frac{\sqrt{1+4x} - \sqrt{1-3x}}{x} \times \frac{\sqrt{1+4x} + \sqrt{1-3x}}{\sqrt{1+4x} + \sqrt{1-3x}}$

$$= \frac{\cancel{1} + 4x - \cancel{1} + 3x}{x(\sqrt{1+4x} + \sqrt{1-3x})}$$

$= \frac{7}{2}$ Ans

(II) $\lim_{x \to 0} \frac{\sqrt{x+a} - \sqrt{a}}{x\sqrt{a(a+x)}} \times \frac{\sqrt{x+a} + \sqrt{a}}{\sqrt{x+a} + \sqrt{a}}$

$$= \frac{\cancel{x} + \cancel{a} - a}{\cancel{x}\sqrt{a(a+x)}(\sqrt{x+a} + \sqrt{a})}$$

$$= \frac{1}{a \cdot 2\sqrt{a}}$$

$$= \frac{1}{2} a^{-3/2}$$

(III) $\boxed{\frac{(1+x^2-1-x)}{(x^2+1-x-1)} \times \sqrt{x^2+1}}$

Q Evaluate

(I) $\lim_{x\to 1}\left(\frac{1}{x^2-1}-\frac{2}{x^4-1}\right)$ $(\infty-\infty)$

Sol:

(II) $\lim_{x\to 1}\left[\frac{(x-2)}{(x^2-x)}-\frac{1}{(x^3-3x^2+2x)}\right]$

Solution :→

(I) $\lim_{x\to 1}\left(\frac{1}{(x+1)(x-1)}-\frac{2}{(x+1)(x-1)(x^2+1)}\right]$

$\lim_{x\to 1}\frac{1}{(x+1)(x-1)}\left[1-\frac{2}{x^2+1}\right]$

$\lim_{x\to 1}\frac{1}{(x+1)\cancel{(x-1)}}\left[\frac{\cancel{x^2-1}}{x^2+1}\right]$

$=\frac{1}{2}$ Ans

x^2-3x+2
$x^2-2x-x+2$
$x(x-2)-1(x-2)$

(II) $\lim_{x\to 1}\left[\frac{(x-2)}{x(x-1)}-\frac{1}{x(x^2-3x+2)}\right]$

$\lim_{x\to 1}\frac{1}{x}\left[\frac{(x-2)}{x-1}-\frac{1}{(x-1)(x-2)}\right]$

$=\lim_{x\to 1}\frac{1}{x(x-1)}\left[(x-2)-\frac{1}{(x-2)}\right]$

$=\frac{x^2-4x+4-1}{x(x-1)}=\frac{x^2-4x+3}{x(x-1)}$

$=\frac{x^2-3x-x+3}{x(x-1)}$

$=\frac{x(x-3)-1(x-3)}{x(x-1)}$

$=\frac{(x-3)}{x}$

$=+2$ Ans

P

Q Evaluate the following limits

(I) $\lim_{\theta \to 0} \frac{\sin 5\theta}{\tan 6\theta}$

(II) $\lim_{x \to 0} \frac{\sin ax}{bx}$

(III) $\lim_{x \to 0} \frac{\cos 7x - \cos 9x}{\cos 3x - \cos 5x}$

(IV) $\lim_{x \to 0} \frac{1 - \cos mx}{x^2}$

(V)

Q (I) $\lim_{x \to \infty} \frac{2x^3 + 2x^2 + 1}{3x^3 + x + 2}$

(II) $\lim_{x \to \infty} \left(\frac{1}{x^2} + \frac{2}{x^2} + \cdots \frac{x}{x^2} \right)$